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The Liouville geometry of $\mathcal{N}=2$ instantons and the moduli of punctured spheres

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Abstract: We propose that the $\mathcal{N}=2$ SYM instanton moduli space maps to the one of punctured spheres, in the framework of the Liouville F-models. This leads to a bilinear recursion relation for the instanton contributions, resembling Hurwitz numbers, and predicts the asymptotic form of the GW invariants. The map is interpreted in the frameworks of the geometric engineering approach to the gauge theory, i.e. the topological A-model, and of the noncritical string theory. We speculate on the extension to the gravitational background and its relation to the uniformization program. Finally, we show an intriguing analogy with the gravitational version of the SU(2) self-dual YM equations where surprisingly the same Hauptmodule of the SW solution appears.

Keywords: Solitons Monopoles and Instantons, String Duality, Supersymmetry and Duality, Differential and Algebraic Geometry.

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Contents

1. Introduction

The Seiberg-Witten (SW) solution [1] of the $\mathcal{N}=2$ super Yang-Mills (SYM) theory is the cornerstone of the today's nonperturbative understanding of the gauge theories. They realized that the monodromy property of the coupling constant around the physical singular points on the moduli space completely determines the prepotential of the low energy effective theory. SW theory is characterized by a Legendre duality [2], whose precise structure is determined by the specific $\mathrm{U}(1)_{\mathcal{R}}$ anomaly fixing the automorphisms of the fundamental domain for the u-plane [4].

There have been along the years many attempts to reconstruct the complete SW solution from a direct instanton computation (for a review see [5]). Originally, the calculation based on the ADHM construction [6] has been problematic. Nevertheless, it has been gradually understood [7, 8, 9] that the instanton amplitudes are topological objects to which localization theorems may be applied. In the end, the all-instanton computation based on the localization technique has been performed by Nekrasov [10, 11]. However, this topological nature of the instantons and their localization properties are hard to see just by looking at the instanton moduli space. The final goal would be to give an algebraic-geometrical formulation of the instanton moduli space and its volume form such that one can directly figure out their localization features. One of the ways in which the instanton moduli space displays its elegant and deep structure is the appearance of recursion relations. It has been known for a long time that the Seiberg-Witten solution obeys some interesting recursion relations among the coefficients of its instanton expansion. Different relations are available in the literature, namely trilinear ones [2], extension to higher rank groups [3], WDVV-like ones [12, 13, 14], linear [15] and contact terms [16, 17]. These recursion relations have been attributed to the underlying topological nature of the instanton moduli space or to the integrable hierarchies hidden in the SW theory.

1.1 Instantons, moduli of punctured spheres and recursion relations

In this paper we study a map between the instanton moduli space of $\mathcal{N}=2$ SYM for SU(2) gauge group and the moduli space of the punctured Riemann spheres. This reconstruction of the instanton moduli space in terms of the moduli space of the punctured sphere is the main results of this paper. Actually, the formulation based on the moduli space of the punctured spheres has an advantage in expressing some algebraic-geometrical feature of the moduli space of instantons. The natural Kähler form on the moduli space of punctured

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sphere is known as the Weil-Petersson (WP) two-form. This WP metric on the punctured spheres not only yields the natural metric on the moduli space, but also reveals a remarkable property which is known as the Wolpert restriction phenomenon. The Wolpert restriction phenomenon guarantees a localization of integral on the boundary of the moduli spaces for some particular integrands.

More precisely, we will show some evidence that the moduli space $\mathfrak{M}_n^I$ of the n-instanton is mapped to the moduli space of the sphere with $4n+2$ punctures, namely

$$\mathfrak{M}_n^I \longrightarrow \overline{\mathcal{M}}_{0,4n+2}\,,$$

by reconstructing the instanton moduli space from SW solution in terms of the moduli space of punctured spheres. In order to establish the precise map we will exploit the Liouville description of the $\overline{\mathcal{M}}_{0,n}$ spaces. A feature of the WP volumes is the appearance of a bilinear recursion relation between them, which is due to the Deligne-Knudsen-Mumford (DKM) compactification of the moduli space together with the Wolpert restriction phenomenon. One way to catch this feature of the moduli space of punctured spheres is to describe it in terms of Liouville theory. It turns out in fact that the classical Liouville action is the Kähler potential for the WP metric. It has been found in [18] that this bilinear recursive structure of the integrals of WP forms on the spaces $\overline{\mathcal{M}}_{0,n}$ is preserved if we slightly deform the WP volume forms ω_n and evaluate instead of the usual WP volume

$$\int_{\overline{\mathcal{M}}_{0,n}} {\omega_n}^{n-3}\,,$$

a deformed volume in which we replace the last insertion with an arbitrary closed two-form

$$\int_{\overline{\mathcal{M}}_{0,n}} {\omega_n}^{n-4} \wedge \omega^F\,.$$

This deformation has been called Liouville F-models. The Liouville F-models are defined as rational intersection theories on $\overline{\mathcal{M}}_{0,n}$ and regarded as a certain universality class of the string theory in a wider sense. This model was originally advocated to describe the nonperturbative aspects of pure quantum Liouville gravity in the continuum language. We will see in this paper that SU(2) SW solution is another example.

The evaluation of such integral defines the expectation value of the two-form ω^F in what we denote as the Liouville background. The benefit of such formulation is that this expectation values obey a master equation. Our master equation refines the original formulation of the Liouville F-models in [18], and all the recursive structures of the integral including its coefficients are now captured by differential operators F_n which characterize the master equation.

The identification of the coefficient of the n-instanton amplitude with an integral over the moduli space of punctured spheres described by the Liouville F-models implies that the instanton moduli space inherits the algebraic-geometrical properties of the former, in particular its recursive structure, related to the DKM compactification. Therefore on the basis of this construction we should expect to find a bilinear recursion relation among instanton coefficients that shares the common properties with the one among the WP volumes. Indeed, we find from the SW solution that the coefficients of the instantons satisfies the following bilinear relation

$$\frac{\mathcal{F}_n}{2\pi i} = \frac{4n-3}{n} \sum_{k=1}^{n-1} e_{k,n} \mathcal{F}_k \mathcal{F}_{n-k} \, . \tag{1.1}$$

This analogy between WP volumes and instantons can be traced back to an apparent surprising coincidence. On the WP side, a nonlinear ODE, implied by the bilinear recursion relation for the WP volumes, can be written as the inverse of a linear differential equation, satisfied by the generating function of such volumes. On the other hand, in the $\mathcal{N}=2$ gauge theory we have a linear differential equation for the periods, the Picard-Fuchs equation (PF), whose inverse is a nonlinear ODE and gives the recursion relation among instanton coefficients, following the other way around. If we now identify the effective gauge coupling constant $\tau(a)$ with the 'coupling constant' for the generating function of the WP volumes, we are led to a map between the $\mathcal{N}=2$ SYM theory and the WP volumes,[1] the latter being described in terms of the classical Liouville theory.

In the literature there are well known cases of explicit maps to $\overline{\mathcal{M}}_{g,n}$ that simplify considerably the calculations. A remarkable example is the Hurwitz space. This is the space of meromorphic functions defining ramified coverings, e.g. of the sphere. This space admits a compactification $\overline{\mathcal{H}}_{g,n}$ consisting of stable meromorphic functions [22]. In particular the projection $\mathcal{H}_{g,n} \longrightarrow \mathcal{M}_{g,n}$ extends to

$$\overline{\mathcal{H}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n} \, . \tag{1.2}$$

As we will see, explicit results are simply obtained thanks to such a map. In particular, calculations simplify considerably in the genus zero case.

1.2 The stringy point of view

Our construction of the instanton amplitudes based on the bilinear recursion relation and Liouville F-models not only provides the reconstruction of the moduli space of instanton in terms of the punctured spheres, but also reveals several connections to the stringy setup for the $\mathcal{N}=2$ SYM theory. In particular, we discuss mostly the connection to the geometric engineering approach and the noncritical string approach in this paper.

Let us begin with the instanton amplitudes again. It is now well-established from the direct instanton calculation that the the integrands localize in the moduli space [7]. This statement has a natural counterpart in our construction of the instanton moduli space in terms of the punctured spheres. Here the DKM boundary of the moduli space plays an significant rôle in evaluating the integral. Furthermore, the bilienar recursion relation also suggests the dynamical selection of the boundary. Namely, we will show that the boundary of the moduli space which contributes the amplitude consists of the divisor which separates the number of punctures by multiple of 4 (+2), which nicely fits the naive expectation that the boundary of the instanton moduli space is given by the collision of two (or more) 'instantons'.

[1] A similar map [19] exists between the $\mathcal{N}=2$ SYM theory and the $\mathcal{N}=1$ SYM theory in the framework of the Dijikgraaf- Vafa (DV) correspondence [20, 21].

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Interestingly enough, the boundary which counts the punctures by four units can be further reduced to that of one unit. In this case, the DKM compactification perfectly works and the nature of the recursion relation is now much like the topological gravity. We would like to interpret the framework of punctured sphere as the geometric engineering approach to the $\mathcal{N}=2$ SYM theory. One can engineer the $\mathcal{N}=2$ SYM theory by considering the topological A-model on a certain noncompact Calabi-Yau manifold and the gauge instanton coefficients are given by the world sheet instantons wrapping some cycles inside the threefold. Here we propose instead a worldsheet approach in which the full topological A-model is considered as a perturbation around the theory which is obtained in the $a \to \infty$ limit, which corresponds to the semiclassical limit in the gauge theory (a as usual denotes the expectation value of the Higgs). The perturbation is achieved by deforming the world sheet CFT. Since the gauge theory prepotential in a flat background is given by the tree level (sphere) free-energy of the A-model, we obtain in this way the instanton coefficients as integrals on the moduli space of n-punctured spheres. This construction can be easily generalized to the presence of a nontrivial gravitational background, namely the graviphoton, whose corrections to the prepotential have recently raised much attention [10, 23].

The direct consequence of our recursion relation and construction of the instanton amplitudes in terms of the moduli space of punctured spheres is the prediction for the asymptotic form of the Gromov-Witten invariants for the local Hirzebruch surface which yields the $\mathcal{N}=2$ SU(2) SYM in a certain limit. Our bilinear recursion relation indeed predicts the rescaled version of the bilinear recursion relation for the asymptotic growth of the Gromov-Witten invariants. Furthermore, our construction of the instanton amplitudes in terms of the intersection theory on the moduli space of punctured spheres states that the asymptotic growth of the Gromov-Witten invariants is calulable as the rational intersection theory on $\overline{\mathcal{M}}_{0,n}$.

On the other hand, the quantum Liouville theory (or $c=0$ noncritical string; for a review see [24]) is very akin to the supersymmetric gauge theory in a sense. The supersymmetric theory depends holomorphically on the parameter, and this property contributes largely to its solvability and so does the Liouville theory. Specifically, the method to derive the correlation functions in the Liouville theory (Goulian-Li [25], Dorn-Otto [26], Zamolodchikov-Zamolodchikov [27]) reminds us of the instanton calculation in the supersymmetric gauge theory (see, e.g. footnote 36 of [24], see also [28]), where we calculate the amplitude when the perturbative expression makes sense and then we analytically continue to the general cases by utilizing the symmetry argument. Actually, there *is* a direct relation between them in the world sheet $\mathcal{N}=2$ super Liouville theory. In such a theory, the Liouville superpotential can be derived from the U(1) vortex condensation, i.e. the instanton effects in the two-dimensional space, from the parent $\mathcal{N}=2$ U(1) gauged linear sigma model [29]. Therefore, the dependence of the cosmological constant in correlation functions is essentially the instanton effect in this perspective. Since it has been conjectured [30, 20] that the bosonic noncritical string theory is deeply related to the topological twist of the $\mathcal{N}=2$ super Liouville theory,[2] the dependence of the cosmological constant in the bosonic

[2]The point is that the $\mathcal{N}=2$ super Liouville theory appears in the description of singular Calabi-Yau

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Liouville theory may have the same origin. In this paper, we push forward this idea and obtain a more direct connection by using our new bilinear recursion relation. We rewrite the instanton contribution to the gauge theory prepotential as the genus expansion of a certain noncritical string theory, which we propose to call 'instanton string theory'. This theory has a striking resemblance to the $c = 0$ Liouville theory in its structure; actually they are in the same universality class of the Liouville F-models. The most intriguing feature is that the amplitude comes only from the boundary of the moduli space much like in the topological gravity [31, 32]. The bilinear recursion relation is nothing but the string equation in this perspective.

1.3 Outline of the paper

The organization of this paper is as follows. In section 2, we review the basic facts on the uniformization property and the moduli space of the punctured spheres. This part consists of the mathematical foundation of the paper. We will show the crucial relation between Liouville theory and the WP volumes and we will introduce the DKM compactification of the moduli space of punctured spheres which, together with the Wolpert restriction phenomenon, leads to the bilinear recursion relation of the WP volumes.

In section 3, we introduce the Liouville F-models and the notion of Liouville background. The former is a certain universality class of the string theory and comes naturally equipped with the bilinear recursive structure. We will present a master equation which provides us a general scheme to treat such bilinear structures in the theory.

In section 4, we discuss the relation between the moduli space of gauge theory instantons and that of the punctured spheres. We propose from the algebraic-geometrical perspective that the former should be described in terms of the latter. We also discuss some stable compactifications of moduli spaces and an example of a map to the moduli space of punctured spheres, namely the map from the space of meromorphic functions on Riemann surfaces, related to the Hurwitz numbers.

In section 5, we describe the instanton coefficients as integrals over the moduli space of the punctured spheres. We introduce a particular Liouville F-model whose master equation predicts the existence of a bilinear recursion relation, which will be found in section 6 starting from the PF equations of SW theory.

In section 6, we present our final form of the Liouville F-model for the SU(2) $\mathcal{N} = 2$ SYM. For this purpose, we derive the bilinear recursion relation hidden in the SW solution. While this relation is anticipated from our discussion so far, here we derive it explicitly with precise coefficients. For the existence of the bilinear recursion relation, we show that the inverse of the PF potential must be at most quadratic. Finally, as a side remark, we point out that if we begin with the bilinear recursion relation *ansatz* with the one-instanton coefficient, we can rederive even the SW solution itself.

In section 7, we discuss the physical interpretation of this bilinear relation from the geometric engineering point of view [33] as well as from the noncritical string theory per-

spaces and the topological B-model, which is describable in terms of matrix models, is related to such Calabi-Yau spaces.

spective. In the former approach, by expressing the instanton amplitudes as integrals on the moduli space of n-punctured spheres, we derive the perturbed CFT expression for the geometric engineering topological A-model. In the latter approach, we show that the gauge coupling constant can be written as the second derivative of a certain noncritical string theory. All these different approaches are based on the underlying Liouville theory.

In section 8 we propose some speculations and future directions. We first discuss the possible dualities among various approach to the SW theory in our view point based on the Liouville geometry. Then we show the extension to the graviphoton background and the relation of our bilinear relation to the underlying recursive structure of the Gromov-Witten invariants. On the relation to the graviphoton background, we point out an intriguing analogy with the self-dual YM equations for the gravitational version of SU(2). Finally, we also speculate on the extension of our results to the higher rank gauge theories.

In section 9 we address some concluding remarks.

In appendix we report the simple proof of Wolpert's restriction phenomenon and the derivation of the Weil-Petersson divisor.

2. Classical Liouville theory and Weil-Petersson volumes

Classical Liouville theory describes the uniformization geometry which is at the heart of the theory of Riemann surfaces (see e.g. [34] for an essential account). In this section we review the basic properties of the classical Liouville theory including its rôle in the description of the geometry of moduli spaces of Riemann surfaces. As we will see, the Liouville action evaluated at the classical solution, S_{cl}, describes the metric properties of the moduli spaces just as the Poincaré metric describes the ones of the Riemann surfaces. More precisely, it turns out that S_{cl} is the Kähler potential for the Weil-Petersson (WP) metric, which, once moduli deformations are described in terms of holomorphic quadratic differentials, can be seen as the 'natural' metric on moduli spaces.[3] This should be compared with the rôle of the Poincaré metric whose logarithm corresponds with its Kähler potential (see the Liouville equation below). Therefore, the Liouville action describes the geometry of both the Riemann surface, providing its natural metric given by the equation of motion, and of their moduli spaces, just coinciding, at its critical point, with the Kähler potential for the WP metric. In this way, roughly speaking, S_{cl} 'transfers' the metric properties of the Riemann surfaces to their moduli space.[4]

2.1 Liouville theory and uniformization of punctured spheres

Here we are mainly interested in the punctured Riemann spheres

$$\Sigma_{0,n} = \widehat{\mathbb{C}}\backslash\{z_1, \ldots, z_n\}\,, \tag{2.1}$$

[3] It turns out that the WP two-form is also in the same cohomological class of the Fenchel-Nielsen two-form (see appendix).

[4] It would be interesting to investigate whether this important property of the Liouville action of generating the metric of both spaces may hold also for other theories.

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where $\widehat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$. Since three punctures can be fixed by a $\mathrm{PSL}(2,\mathbb{C})$ transformation, different complex structures may arise only for $n \geq 3$. Let us introduce the moduli space of the punctured Riemann spheres

$$\mathcal{M}_{0,n} = \{(z_1,\ldots,z_n) \in \widehat{\mathbb{C}}^n | z_j \neq z_k \,\mathrm{for}\, j \neq k\}/\mathrm{Symm}(n) \times \mathrm{PSL}(2,\mathbb{C})\,, \tag{2.2}$$

where $\mathrm{Symm}(n)$ acts by permuting $\{z_1,\ldots,z_n\}$ whereas $\mathrm{PSL}(2,\mathbb{C})$ acts as a linear fractional transformation. We use the latter transformations to set $z_{n-2} = 0$, $z_{n-1} = 1$ and $z_n = \infty$, so that we have

$$\mathcal{M}_{0,n} \cong V^{(n)}/\mathrm{Symm}(n)\,, \tag{2.3}$$

where

$$V^{(n)} = \{(z_1,\ldots,z_{n-3}) \in \mathbb{C}^{n-3} | z_j \neq 0,1\ ; z_j \neq z_k\,, \qquad \mathrm{for}\ j \neq k\}\,. \tag{2.4}$$

A fundamental object in the theory of Riemann surfaces is the uniformizing mapping

$$J_{\mathbb{H}} : \mathbb{H} \longrightarrow \Sigma_{0,n}\,, \tag{2.5}$$

with $\mathbb{H} = \{w | \mathrm{Im}\, w > 0\}$ the upper-half plane. The Poincaré metric $ds^2 = \frac{|dw|^2}{(\mathrm{Im}\, w)^2}$ on $\mathbb{H}$, is the metric of constant scalar curvature -1. Since $w = J_{\mathbb{H}}^{-1}(z)$, this induces on the Riemann surface the metric $ds^2 = e^{\varphi}|dz|^2$, where

$$e^{\varphi} = \frac{|J_{\mathbb{H}}^{-1\prime}|^2}{(\mathrm{Im}\, J_{\mathbb{H}}^{-1})^2}\,. \tag{2.6}$$

The fact that the metric has constant curvature -1 is the same of the statement that φ satisfies the Liouville equation

$$\partial_{\bar{z}}\partial_z \varphi = \frac{e^{\varphi}}{2}\,. \tag{2.7}$$

An important object is the Liouville stress tensor

$$T(z) = \left\{J_{\mathbb{H}}^{-1}(z), z\right\} = \varphi_{zz} - \frac{1}{2}\varphi_z^2\,, \tag{2.8}$$

where $\{f(z),z\} = f'''/f' - \frac{3}{2}(f'/f'')^2$ is the Schwarzian derivative. In the case of the punctured Riemann spheres we have

$$T(z) = \sum_{k=1}^{n-1}\left[\frac{1}{2(z-z_k)^2} + \frac{c_k}{z-z_k}\right]\,, \tag{2.9}$$

where the *accessory parameters* $c_1,\ldots,c_{n-1}$ are functions on $V^{(n)}$. They satisfy the two conditions

$$\sum_{j=1}^{n-1} c_j = 0\,, \qquad \sum_{j=1}^{n-1} z_j c_j = 1 - \frac{n}{2}\,. \tag{2.10}$$

Let us write down the Liouville action [35]

$$S^{(n)} = \lim_{r\to 0}\left[\int_{\Sigma_{0,n}^r} (\partial_z\varphi\partial_{\bar{z}}\varphi + e^{\varphi}) + 2\pi(n\mathrm{log}r + 2(n-2)\mathrm{log}|\mathrm{log}r|)\right]\,, \tag{2.11}$$

where

$$\Sigma_{0,n}^{r}=\Sigma_{0,n}\backslash\left(\bigcup_{i=1}^{n-1}\{z||z-z_i|<r\}\cup\{z||z|>r^{-1}\}\right). \tag{2.12}$$

It turns out that the accessory parameters are strictly related to $S^{(n)}$ evaluated at the classical solution. More precisely, we have the Polyakov conjecture (see also [36] for a recent discussion)

$$c_k=-\frac{1}{2\pi}\partial_{z_k}S_{cl}^{(n)}\,, \tag{2.13}$$

which has been proved in [35]. Furthermore, it turns out that S_{cl} is the Kähler potential for the Weil-Petersson two-form [35]

$$\omega_{WP}^{(n)}=\frac{i}{2}\bar\partial\partial S_{cl}^{(n)}=-i\pi\sum_{j,k=1}^{n-3}\frac{\partial c_k}{\partial\bar z_j}d\bar z_j\wedge dz_k\,, \tag{2.14}$$

showing that Liouville theory describes the geometry of the moduli space of Riemann surfaces. As a result, note that even in critical string theory the classical Liouville action appears in the string measure on moduli space.

2.2 Deligne-Knudsen-Mumford compactification

We now consider the DKM stable compactification $\overline{V}^{(n)}$ of the moduli space $V^{(n)}$ [37]. Let us consider a nontrivial cycle of $\Sigma_{0,n}$. In shrinking it the complex structure of $\Sigma_{0,n}$ changes and so we move around $V^{(n)}$. In the limit, when the cycle is completely shrunk, the degenerate surface does not belong to $V^{(n)}$. A similar situation arises when one tries to collide two punctures. In the stable compactification there appears a long tiny neck separating them with the final configuration corresponding to two surfaces glued by a node (a double puncture), both having a number of punctures ≥ 3, so that the Riemann surfaces involved are always negatively curved. Once the node is removed, we obtain two Riemann spheres with $k+2$ and $n-k$ punctures, where the value of $k=1,\ldots,n-3$ depends on the number of encircled punctures. This sort of exclusion principle, according to which punctures never collide,[5] is essentially a consequence of the Gauss-Bonnet theorem.

The DKM boundary of $V^{(n)}$ consists of the moduli spaces corresponding to such degenerate configurations. Since the dimension of $V^{(n)}$ is $n-3$, one sees that the boundary has codimension one. Furthermore, while the number of punctures increased by two, the total Euler characteristic is unchanged.

A basic property of the DKM compactification is that it has a clear recursive structure

$$\begin{array}{lll} & & \overline{V}^{(k+2)}\longrightarrow\overline{V}^{(j+2)}\times\overline{V}^{(k+2-j)}\ldots\\ & \nearrow & \\ \overline{V}^{(n)}\longrightarrow\overline{V}^{(k+2)}\times\overline{V}^{(n-k)} & & \\ & \searrow & \\ & & \overline{V}^{(n-k)}\longrightarrow\overline{V}^{(j+2)}\times\overline{V}^{(n-k-j)}\ldots \end{array}$$

[5]Note that each of the two punctures of the node belong to different surfaces and in the Poincaré metric their distance is infinity.

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This recursive structure already suggests that for some suitable forms one may get the localization property

$$\int_{\overline{V}^{(n)}} \sim \sum_{k=1}^{n-3} c_k \int_{\overline{V}^{(k+2)}} \int_{\overline{V}^{(n-k)}} , \tag{2.15}$$

leading to bilinear recursion relation. This would imply that the generating function for such integrals satisfy nonlinear differential equations. Therefore from the structure of the boundary we may get an exact resummation, that is a 'nonpertubative result'. Remarkably, as we will see, in several cases including $\mathcal{N} = 2$ SYM and WP volumes, such equations are essentially the inverses of linear ones.

2.3 WP volumes recursion relation

In order to be formalized, the above description needs a counting of the number of times a given component $\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}$ appears in the boundary of $\overline{V}^{(n)}$. Of course, this is given by the different ways we may encircle a fixed number of punctures. Thus, we introduce the divisors $D_1, \ldots, D_{[n/2]-1}$ which are subvarieties of codimension one, that is they have real dimension $2n-8$. Each D_k represents, with the above combinatorics, surfaces that split, upon removal of the node, into two Riemann spheres with $k+2$ and $n-k$ punctures. In particular, D_k consists of $C(k)$ copies of $\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}$ where

$$C(k) = \binom{n}{k+1} , \tag{2.16}$$

$k = 1, \ldots, (n-3)/2$, for n odd. In the case of even n the unique difference is for $k = n/2-1$, for which we have

$$C\left(\frac{n}{2} - 1\right) = \frac{1}{2}\binom{n}{n/2} . \tag{2.17}$$

It turns out that the image of the divisors D_k's provides a basis in $H_{2n-8}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$. The DKM boundary simply consists of the union of the divisors D_k's, that is

$$\partial\overline{V}^{(n)} = \bigcup_{j=1}^{[n/2]-1} D_j . \tag{2.18}$$

For future purposes it is convenient to extend the range of the index of D_j by setting

$$D_k = D_{n-k-2} , \qquad k = 1, \ldots, n-3 . \tag{2.19}$$

Let us consider the WP volume[6]

$$\mathrm{Vol}_{WP}\left(\overline{\mathcal{M}}_{0,n}\right) = \frac{1}{n!(n-3)!} \int_{\overline{V}^{(n)}} \omega_{WP}^{(n)\ n-3} . \tag{2.20}$$

As we will see, the volumes are rational numbers up to powers of π, so we set

$$V_n = \pi^{2(3-n)}(n-3)! \mathrm{Vol}_{WP}\left(\overline{V}^{(n)}\right) , \tag{2.21}$$

[6] Note that

$$\mathrm{Vol}_{WP}\left(\overline{\mathcal{M}}_{0,n}\right) = \frac{1}{n!} \mathrm{Vol}_{WP}(\overline{V}^{(n)}) .$$

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and will consider the rescaled WP two-form

$$\omega_n = \frac{\omega_{WP}^{(n)}}{\pi^2}\,, \tag{2.22}$$

as it will lead to rational cohomology.

In a remarkable paper [38], Zograf calculated such volumes recursively. This construction is simple and elegant. It is instructive to illustrate the main features leading to his recursion relation.

1. The first step is to note that since the divisors D_k's provide a basis in $H_{2n-8}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$, the WP two-form ω_n has a Poincaré dual given by a linear combination of the D_k's,[7] so that V_n reduces to an integral on the boundary of $\overline{V}^{(n)}$.

2. The recursive structure of the DKM boundary, given by the structure of the D_k's then implies that V_n is expressed as a sum of integrals on $\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}$. Thus we start seeing the recursive structure

$$\int_{\overline{V}^{(n)}} {\omega_n}^{n-3} = \sum_{k=1}^{n-3} c_k \int_{\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}} \rho_k\,, \tag{2.23}$$

where c_k are some combinatorial factors.

3. The last step is the observation that the form ρ_k is expressed just in terms of the WP two-forms of $\overline{V}^{(k+2)}$ and $\overline{V}^{(n-k)}$. This is due to the Wolpert restriction phenomenon [39], according to which the restriction of the WP two-form ω_n on each component $\overline{V}^{(k+2)}$ of the DKM boundary is in the same cohomological class of ω_{k+2}. More precisely, we have[8]

Theorem. *(Wolpert [39]). Let i denote the natural embedding*

$$i : \overline{V}^{(m)} \to \overline{V}^{(m)} \times * \to \overline{V}^{(m)} \times \overline{V}^{(n-m+2)} \to \partial\overline{V}^{(n)} \to \overline{V}^{(n)}\,, \tag{2.24}$$

for $n > m$, where $*$ is an arbitrary point in $\overline{V}^{(n-m+2)}$. Then

$$[\omega_m] = i^*\,[\omega_n]\,, \qquad n > m\,. \tag{2.25}$$

We report in the appendix a simple proof of this theorem. This theorem implies that (2.23) becomes

$$\int_{\overline{V}^{(n)}} {\omega_n}^{n-3} = \sum_{k=1}^{n-3} \tilde{c}_k \int_{\overline{V}^{(k+2)}} {\omega_{k+2}}^{k-1} \int_{\overline{V}^{(n-k)}} {\omega_{n-k}}^{n-k-3}\,. \tag{2.26}$$

The above is the essential account of Zograf's recursion relation [38]

$$V_n = \frac{1}{2}\sum_{k=1}^{n-3} \frac{k(n-k-2)}{n-1} \binom{n}{k+1} \binom{n-4}{k-1} V_{k+2} V_{n-k}\,, \tag{2.27}$$

where $n \geq 4$ and $V_3 = 1$.

[7] The derivation of the Poincaré dual to ω_{WP} is reported in appendix.

[8] For an excellent updating on the WP metric see [40].

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2.4 The equation for the WP volumes generating function

Zograf's recursion relation originated a series of interesting results in the framework of quantum cohomology [41]. The first observation is that a rescaling of V_k simplifies Zograf's recursion relation considerably, that is by setting [18]

$$a_k = \frac{V_k}{(k-1)[(k-3)!]^2}\,, \tag{2.28}$$

for $k \geq 3$, eq. (2.27) becomes

$$a_3 = \frac{1}{2}\,, \qquad a_n = \frac{1}{2}\frac{n(n-2)}{(n-1)(n-3)}\sum_{k=1}^{n-3} a_{k+2}a_{n-k}\,, \qquad n \geq 4\,. \tag{2.29}$$

This is a useful step because we can introduce the generating function of the WP volumes:

$$g(x) = \sum_{k=3}^{\infty} a_k x^{k-1}\,, \tag{2.30}$$

and then derive the associated nonlinear differential equation [18]

$$x(x-g)g'' = x{g'}^2 + (x-g)g'\,. \tag{2.31}$$

The recursion relation (2.29) has been crucial in formulating a nonperturbative model of Liouville quantum gravity as a Liouville F-model [18] (see also [42]), which we will review in section 3. In particular, this formulation has been obtained as a deformation of the WP volumes. Furthermore, it has been argued that the relevant integrations on the moduli space of higher genus Riemann surfaces can be reduced to integrations on the moduli space of punctured Riemann spheres (see also [43, 44]).

It has been shown by Kaufmann, Manin and Zagier [43] that this nonlinear ODE is essentially the inverse of a linear one. More precisely, defining $g = x^2\partial_x x^{-1}h$, one has that (2.31) implies

$$xh'' - h' = (xh' - h)h''\,. \tag{2.32}$$

Differentiating (2.32) we get

$$yy'' = xy'^3\,, \tag{2.33}$$

where $y = h'$. Then, interchanging the rôles of x and y, one can transform (2.33) into the Bessel equation

$$\left(\partial_y^2 + \frac{1}{y}\right)x = 0\,. \tag{2.34}$$

One known solution of this equation is a modified Bessel function, which is exactly the inverse of the generating function of the WP volumes. More precisely, we have [43]

$$x(y) = -\sqrt{y}J_0'(2\sqrt{y})\,, \tag{2.35}$$

where

$$J_0(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z^2}{4}\right)^n\,, \tag{2.36}$$

so that

$$y(x) = \sum_{n=3}^{\infty} \frac{V_n}{(n-2)!(n-3)!} x^{n-2} \qquad \Longleftrightarrow \qquad x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!(m-1)!} y^m \,. \tag{2.37}$$

The modified Bessel function is convergent in all the complex plane, but not its inverse. The first zero of the derivative $x'(y) = J_0(2\sqrt{y})$ tells us when the inverse stops to converge. One can calculate the asymptotic form of the WP volumes in this way.

2.5 A surprising similarity

There is a surprising similarity between the general structure involved in the derivation of the WP volumes and the one for the instantons in $\mathcal{N} = 2$ SYM theory for SU(2) gauge group [2]. In SW theory one starts from the linear differential equation satisfied by $a(u)$ and inverts it to a nonlinear differential equation satisfied by $u = \mathcal{G}(a) = \sum_{k=0}^{\infty} a^{2-4k} \mathcal{G}_k$. This in turn implies the recursion relation for the instanton coefficients $\mathcal{G}_k$ (related to the ones for the prepotential by $\mathcal{G}_k = 2\pi i k \mathcal{F}_k$). In the case of WP volumes, one starts from the opposite side with respect to SW theory: first directly evaluates the recursion relation making use of the DKM compactification and of the Wolpert restriction phenomenon, and arrives to the nonlinear differential equation (2.31) which should be compared with the one in $\mathcal{N} = 2$ SYM [2]

$$(1 - \mathcal{G}^2)\mathcal{G}'' + \frac{a}{4}\mathcal{G}'^3 = 0 \,. \tag{2.38}$$

What is crucial is that, like (2.38), also (2.31) is essentially the inverse of a linear ODE (2.34).

This stringent analogy strongly suggests the possibility to reobtain the $\mathcal{N} = 2$ SYM results just starting from the point where Zograf started, that is by directly evaluating the recursion relation using algebraic-geometrical techniques in instanton theory. This would be possible in the framework of the Liouville F-models introduced in [18]. In this respect we note that, besides the recursive nature of the DKM compactification and of the Wolpert restriction phenomenon, two steps leading to localization phenomena, one of the observations was that the original recursion relation is actually reduced to the simple form (2.29). Furthermore, a key step was the observation that the recursive structure obtained by Zograf admits an important generalization. Namely, in [18] it was observed that the recursive structure arising in the evaluation of the integrals $\int_{\overline{\mathcal{M}}_{0,n}} {\omega_n}^{n-3}$ persists even if one considers the replacement

$${\omega_n}^{n-3} \longrightarrow {\omega_n}^{n-4} \wedge \omega^F \,, \tag{2.39}$$

for suitable closed two-forms ω^F. This led to the nonperturbative formulation of Liouville quantum gravity in the continuum [18].

3. The Liouville F-models and the master equation

In this section, we consider the Liouville F-models introduced in [18]. The Liouville F-models are defined as a certain universality class of the string theory given by the integration over the moduli space of the punctured spheres. For example as it was discussed in [18], the $c = 0$ noncritical string theory is in this class. Later we will see that also the SW instanton contributions are in this category.

The general motivation in [18] to introduce the Liouville F-models is based on the following observation. A general theory of string can be represented as a summation on the genus g, of integrals on the moduli space of Riemann surfaces $\overline{\mathcal{M}}_g$. Usually such integrals are technically impossible to evaluate. However, for some theories, these integrals may actually be performed. In particular, when the theory leads to a localization in the measure, one expects contributions from the DKM boundary of $\overline{\mathcal{M}}_g$, which is composed of the union of copies of the spaces $\overline{\mathcal{M}}_{g-1,2}$ and $\overline{\mathcal{M}}_{g-k,1} \times \overline{\mathcal{M}}_{k,1}$, $k = 1, \ldots, g-1$. On the other hand, depending on the structure of the integrand, it may happen that also each one of the integrals on these lower dimensional spaces obtains contributions only from the relative boundary. Iterating this would lead to a final configuration corresponding to integrals on the moduli space of Riemann surfaces without handles and with punctures, which is obtained when all the cycles around the handles are pinched. In this case, the initial integration on $\overline{\mathcal{M}}_g$ reduces to a sum of products of integrals on the moduli space of punctured spheres $\overline{\mathcal{M}}_{0,n}$. As a result, after a complete shrinking, the relevant geometry of the string genus expansion collapses to the one of $\overline{\mathcal{M}}_{0,n}$, that is

$$\{\overline{\mathcal{M}}_g\} \qquad \Longrightarrow \qquad \{\overline{\mathcal{M}}_{0,n}\}\,, \tag{3.1}$$

and for such theories one has to evaluate integrals of the kind

$$\int_{\overline{\mathcal{M}}_{0,n}} \rho\,, \tag{3.2}$$

for some form ρ. Furthermore, the metric properties of the moduli space that become the natural measure are described in terms of the WP two-form, which in turn has the classical Liouville action as Kähler potential. In this respect, we note that the appearance of the classical Liouville action suggests that for such theories there is a relation between localization properties of the measure and the semiclassical approximation (as it may happen in some noncritical string theories). Thus, one expects that the theory should be expressed as

$$Z_n^F = \int_{\overline{V}^{(n)}} {\omega_n}^{n-3} e^{-S_F}\,, \tag{3.3}$$

where S_F is the effective action on the moduli, that is the remnant of the initial quantum action.

3.1 The Liouville background

If one has the effective action S_F which defines a closed form $\omega^F = e^{-S_F}\omega_n$ up to exact terms,[9] a special feature emerges: the amplitudes Z_n^F show a recursive structure and this distinguishes among others the Liouville F-models we define in the following. In this respect, we mentioned that the replacement (2.39) leads to new recursion relations.

[9] Actually, the condition can be weaker in our purpose: the equality should hold only after the integration wedged by ω_n^{n-4} over the moduli space of the punctured spheres.

Therefore, as observed in [18], we can keep the recursion relation property of the WP volume form not only for ω_n^{n-3}, but also for other $(2n-6)$-forms. To understand this, note that if we make the replacement (2.39) the volumes V_n are replaced by

$$[\omega_n^{n-4} \wedge \omega^F] \cap \left[\overline{V}^{(n)}\right] = [\omega_n]^{n-4} \cap [D^F] \,, \tag{3.4}$$

where $\cap$ is the topological cup product and D^F is the Poincaré dual to $[\omega^F]$. Expanding the latter in terms of the homological basis, (3.4) becomes

$$\sum_k b_k \, [\omega_n]^{n-4} \cap \left[\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}\right] = \sum_k b_k \, [\omega_{k+2} + \omega_{n-k}]^{n-4} \cap \left[\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}\right], \tag{3.5}$$

where we used the Wolpert restriction phenomenon we review in the appendix. Since the unique contribution comes from the terms in the binomial expansion having the correct dimension with respect to $V^{(k+2)}$ and $V^{(n-k)}$, we have

$$\sum_k b_k \, [\omega_n]^{n-4} \cap \left[\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}\right] = \sum_k b_k \binom{n-4}{k-1} V_{k+2} V_{n-k} \,. \tag{3.6}$$

Thus, for suitable ω^F's one has recursive relations.

The theories we consider are the ones with an effective action S_F such that

$$Z_n^F = \langle \omega^F \rangle_n \,, \tag{3.7}$$

where

$$\langle \sigma \rangle_n \equiv \frac{1}{n!} \int_{\overline{V}^{(n)}} {\omega_n}^{n-4} \wedge \sigma \,. \tag{3.8}$$

This notation synthesizes some of the peculiar properties enjoyed by the moduli space of punctured spheres and of its Liouville geometry. To explain this, first recall that since

$$\omega_n = \frac{i}{2\pi^2} \overline{\partial} \partial S_{cl}^{(n)} \,, \tag{3.9}$$

we see that the notation (3.8) defines the evaluation of a given two-form in the Liouville background. Furthermore, we have seen that for suitable ω^F's, the substitution (2.39) preserves the recursive structure of the integrals. In this respect the notation involves all these information, in particular in doing the integration one uses the DKM compactification and the Wolpert restriction phenomenon, whereas ω_n provides the Liouville background.

In order to define the divisor D^F, we first observe that to express the theory in terms of integrals on $\overline{V}^{(n)}$, we need its recursive structure rather than the specific values of the WP volumes. Therefore we introduce the normalized divisors

$$\mathcal{D}_k = \frac{D_k}{\langle \sigma_k \rangle_n} \,, \tag{3.10}$$

where $[\sigma_k]$ is the Poincaré dual to D_k, so that[10]

$$\langle \mu_k \rangle_n = 1 \,, \tag{3.11}$$

with $[\mu_k]$ the dual of $\mathcal{D}_k$.

[10] Note that $(\langle \omega \rangle_j \equiv \langle \omega_j \rangle_j)$

$$\langle \sigma_k \rangle_n = \frac{(n-4)!(k+2)(n-k)}{(n-k-3)!(k-1)!} \langle \omega \rangle_{k+2} \langle \omega \rangle_{n-k} \,.$$

3.2 Intersection theory and the boostrap

From the above discussion, it is clear that we can regard this formulation as a deformation of the WP volumes. In particular, recalling that each divisor D_k is composed of copies of $\overline{V}_{k+2} \times \overline{V}_{n-k}$, labelled by two different subscripts, as will become clear in the following, a general deformation is conveniently expressed by introducing two parameters s and t. Thus, we introduce the two-form $\eta(s,t)$ whose class is defined by its Poincaré dual

$$D_\eta = \sum_{k=1}^{n-3} e^{(k+2)s} e^{(n-k)t} \langle \omega^F \rangle_{k+2} \langle \omega^F \rangle_{n-k} \mathcal{D}_k \,. \tag{3.12}$$

To understand the nature of the class $[\eta(s,t)]$, note that since the theories we are considering share the recursive geometry of $\overline{\mathcal{M}}_{0,n}$, the divisor D_η includes the physical information on the 'correlators' on the spaces $\overline{\mathcal{M}}_{0,k+2} \times \overline{\mathcal{M}}_{0,n-k}$. Essentially, once the theory is defined, we should evaluate the (rational) intersection of the divisor associated to ω^F with the DKM boundary. Even if we do not know the explicit expression of such ω^F, we can use the iterative structure of $\overline{\mathcal{M}}_{0,n}$ to define ω^F on $\overline{\mathcal{M}}_{0,n}$ in terms of the ones defined on $\overline{\mathcal{M}}_{0,k+2} \times \overline{\mathcal{M}}_{0,n-k}$. It follows that we only need the first correlator, that is $\langle \omega^F \rangle_3$, as initial condition, to recover the full tower of ω^F's. This means that we compute ω^F on $\overline{\mathcal{M}}_{0,n}$, and therefore the '$n$-point function' $\langle \omega^F \rangle_n$, starting from the 'three-point function' $\langle \omega^F \rangle_3$. This sort of boostrap is a direct consequence of the recursive structure of the DKM compactification and of the Wolpert restriction phenomenon.

3.3 The master equation and bilinear relations

As we have seen so far, the bilinear recursion relation for the WP volume (2.29) has a natural generalization:

$$Z_n^F = \sum_{k=1}^{n-3} F_n(k+2, n-k) Z_{k+2}^F Z_{n-k}^F \,. \tag{3.13}$$

Due to the structure of the DKM boundary, the coefficients $F_n(k+2, n-k)$ defining the general bilinear recursion relation that can be obtained by evaluating suitable integrations on $\overline{\mathcal{M}}_{0,n}$ can be chosen to be symmetric under $k+2 \to n-k$, so that in general

$$F_n(k+2, n-k) = \sum_j h_j(n)(k+2)^j (n-k)^j \,. \tag{3.14}$$

The coefficients F_n are however redundant, in particular we are interested in the class of equivalence of recursion relations that differ each other only by a trivial rescaling of the terms. A natural choice to select representatives of such equivalence class is to set[11]

$$h_j = 0 \,, \qquad \forall j < 0 \,, \qquad \text{and} \qquad h_0 \neq 0 \,. \tag{3.15}$$

The general Liouville F-models are defined by the generating function for the Z_n^F where $[\omega^F]$ is given by the master equation

$$[\omega^F] = F_n(\partial_s, \partial_t)[\eta_0] \,, \tag{3.16}$$

[11] To be precise, this does not uniquely fix h_0. However, this is irrelevant for our purpose.

where $\eta_0 \equiv \eta(0,0)$. This equation fixes the recursion relations once the initial condition, that is the three-point function $\langle \omega^F \rangle_3$, is given. In terms of the Poincaré dual, the master equation reads

$$D^F = F_n(\partial_s, \partial_t) D_{\eta_0} \,. \tag{3.17}$$

Evaluating the master equation (3.16) on the Liouville background

$$\langle \omega^F \rangle_n = F_n(\partial_s, \partial_t) \langle \eta_0 \rangle_n \,, \tag{3.18}$$

and by (3.11)

$$\langle \omega^F \rangle_n = \sum_{k=1}^{n-3} F_n \langle \omega^F \rangle_{k+2} \langle \omega^F \rangle_{n-k} \,, \tag{3.19}$$

which nicely summarizes the recursive properties of $\overline{\mathcal{M}}_{0,n}$. In this respect we note that such a recursion resembles a sort of generalization to $\overline{\mathcal{M}}_{0,n}$ of the Riemann bilinear relations. Actually, both eq. (3.19) and the Riemann bilinear relations express the volume form as a linear combination of a the product of two lower dimensional integrals.

In the previous analysis, we used the recursive properties of the DKM compactification to introduce the concept of F-models defined in terms of a suitable set of cohomological classes $[\omega^F]$ that, evaluated on the Liouville background, enjoys the recursive properties. Besides the DKM compactification, it is just the Liouville background that, due to the Wolpert restriction phenomenon, leads to recursion relations.

3.4 Pure Liouville quantum gravity

The generating functions for the Liouville F-models are [18]

$$\mathcal{Z}^{F,\alpha}(x) = x^{-\alpha} \sum_{n=3}^{\infty} x^n \langle \omega^F \rangle_n \,, \tag{3.20}$$

which are classified by α, F_n and $\langle \omega^F \rangle_3$. The Liouville F-models include pure quantum Liouville gravity, which can therefore be formulated in the continuum in terms of deformation of the WP volumes whose geometry is described by the classical Liouville theory [18].

One of the distinguishing features of pure quantum Liouville gravity is that, like in the case of WP volumes, we have $h_j = 0$, $\forall j \neq 0$, and the master equation takes the simple form

$$[\omega^F] = h_0 [\eta_0] \,. \tag{3.21}$$

Evaluating this equation on the Liouville background gives

$$\langle \omega^F \rangle_n = h_0 \langle \eta_0 \rangle_n = h_0 \sum_{k=1}^{n-3} \langle \omega^F \rangle_{k+2} \langle \omega^F \rangle_{n-k} \,, \tag{3.22}$$

$n \geq 4$, where in the case of pure Liouville gravity [18]

$$h_0 = \frac{3}{(12-5n)(13-5n)} \,, \qquad \langle \omega^F \rangle_3 = -\frac{1}{2} \,. \tag{3.23}$$

Setting $\mathcal{Z}(t) = \mathcal{Z}^{F,\frac{12}{5}}(t^5)$, one sees that from the master equation it follows that the specific heat of pure Liouville quantum gravity corresponds to the series of the two-form ω^F evaluated on the Liouville background

$$\mathcal{Z}(t) = t^{-12} \sum_{n=3}^{\infty} t^{5n} \langle \omega^F \rangle_n \,, \tag{3.24}$$

which in fact satisfies the Painlevé I

$$\mathcal{Z}^2 - \frac{1}{3}\mathcal{Z}'' = t \,, \tag{3.25}$$

with initial conditions $\mathcal{Z}(0) = \mathcal{Z}'(0) = 0$. Recalling that

$$\omega_n = \frac{i}{2\pi^2} \bar{\partial}\partial S_{cl}^{(n)} \,, \tag{3.26}$$

we see that the formulation in the continuum of pure Liouville quantum gravity is expressed in terms of the Liouville action evaluated at the classical solution.

Since this solution is not the standard perturbative solution, we briefly discuss its nature before concluding this subsection. The physical consequences of the model have been investigated in [42]. In particular, it turns out that the model corresponds to the quantum Liouville theory with Einstein-Hilbert action having an imaginary part $\pi/2$. In other words. Eq. (3.24) corresponds to introducing a Θ-vacuum structure in the genus expansion [42]

$$\mathcal{F}(t) = \sum_{g=0}^{\infty} \int_{Met_g} \mathcal{D}h e^{-S(h) + i\frac{\Theta}{2\pi}\int_\Sigma R\sqrt{h}} = \sum_{g=0}^{\infty} (-1)^{1-g} \int_{Met_g} \mathcal{D}h e^{-S(h)} \,, \qquad \Theta = \frac{\pi}{2} \,, \tag{3.27}$$

where the specific heat is defined as $\mathcal{Z}(t) = -\mathcal{F}''(t)$. The effect of the Θ-term is to convert the expansion into a series of alternating signs which is Borel summable.

An important point is that the specific heat of the model has a physical behavior. According to standard thermodynamics, if one defines, following [45], the 'specific heat' as the second derivative of the free-energy, it should be negative. In [42] it has been shown that the specific heat is negative for all $t > 0$, whereas in the standard choice [45] for the boundary condition in the asymptotic expansion is always positive for sufficiently large t. It seems that this choice is made in order to avoid an apparently unphysical behaviour such as the alternating signs of the asymptotic series.[12] However alternating signs in perturbation theory cannot have a nonpertubative quantum field theoretical meaning. In other words this 'unphysical behaviour' is only apparent, that is an effect of the perturbation expansion. What makes sense are the nonperturbative results which are in complete agreement with basic physical principles. Thus the results of the model agree with standard thermodynamics and the theory is Borel summable.

As emphasized in [42], the rôle of Θ-vacua is suggestive for string theory in general. This aspect is related to the structure of the moduli space and to unitarity problems. To understand the relation between unitarity and the structure of moduli space one should

[12] Note that series with coefficients having alternating signs can be obtained just by changing the point of the expansion.

consider that degenerated surfaces correspond to Feynman diagrams. The rôle of Θ-vacua should follow from a Feynman diagram analysis like applied to the string path-integral at the boundary of moduli spaces. We also notice that the presence of Θ-vacua should improve the convergence of the perturbation theory of critical strings. In other words one should expect that string perturbation theory with Θ-vacua converges.

4. Instanton moduli spaces and $\overline{\mathcal{M}}_{0,n}$

Now let us turn to our main theme, the Liouville geometry of the $\mathcal{N}=2$ instantons and moduli space of punctured spheres. As we have briefly seen in the introduction, the derivation of the prepotential in SW theory from the direct instanton calculation has been steadily developed (for a review see [5]). Originally, despite the existence of the exact solution by SW, the direct instanton calculation based on the ADHM construction [6] has been difficult to perform. Nevertheless, it has been gradually understood [7, 8, 9] that the instanton amplitudes are topological objects to which the localization theorem may be applied. To fully utilize the technique of the localization, some desingularization of the instanton moduli space is necessary, and interesting results have been obtained by Hollowood in [9] by introducing the noncommutative geometry (for a more mathematical treatment, see [46] and references therein). Finally the all-instanton solution from the direct instanton calculation based on the localization technique is presented by Nekrasov [10, 11], where by using the so-called Ω background, the enumerative evaluation of the localized integral becomes possible.[13]

In spite of several proposals, there remain some open questions, for example the one concerning the compactification. In particular, even if the above results provide a major step towards a better understanding of the instanton moduli space of $\mathcal{N}=2$ SYM theory, it remains to be uncovered whether the instanton moduli space might admit a more algebraic-geometrical description. The first step in such a direction has been considered in [49, 50, 8]. The main idea there was to reconstruct the instanton moduli space from the known SW solution. In particular, the strong analogy between Zograf's derivation of the recursion relation, which leads to a linear differential equation and the derivation of the instanton recursion relation from the linear differential equation, suggested a possibility to construct a space whose structure and volume form directly implies the instanton recursion relation. However, in such a construction they have used the original recursion relation, which is a trilinear one.

There are several results in algebraic-geometry just due to the choice of a good compactification of a given moduli space. For example, as we have seen, the DKM compactification, together with the Wolpert restriction phenomenon, are the two main ingredients leading to Zograf's recursion relation. On the other hand, we have seen that a suitable deformation of the WP volume form still leads to recursion relations. Therefore, the strong similarities we have seen in section 2.5 between the derivation of WP recursion relation and the one for the $\mathcal{N}=2$ instantons may actually be a consequence of a formulation of

[13] Recently this method has been extended to other gauge groups in [47, 48].

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instanton theory based on the moduli space of punctured spheres. This would lead to introducing the DKM stable compactification and therefore to quantum cohomology for $\mathcal{N}=2$ SYM theory, a fact that may not be considered a surprise since quantum cohomology has several features of gauge theories.

Besides the above-mentioned similarity between WP volumes and $\mathcal{N}=2$ instantons, we also note the theorem (2.14) for the WP two-form

$$\omega_n = \frac{i}{2\pi^2}\overline{\partial}\partial S_{cl}^{(n)}\,, \tag{4.1}$$

which relates the Kähler potential of the WP metric to the classical Liouville action. This is reminiscent of the fact that for the hyper Kähler metric of the instanton moduli space, we have [51]

$$K(\omega_I^{(k)}) = -\frac{1}{4}\int d^4x x^2 \,\mathrm{Tr}\, F_{mn}^{(k)2}\,, \tag{4.2}$$

where $K(\omega_I^{(k)})$ is the hyper Kähler potential for the natural volume of the instanton moduli space which is deduced from the functional integral, and we evaluate the integral by substituting the classical k instanton solution just as in (4.1).

4.1 Stable compactification and the bubble tree

The program of using the stable compactification, and therefore quantum cohomology, for the instantons has been already considered in the literature (see for example [52]). The important quantity here is the moduli space of stable maps. The original proposal of a similar compactification for moduli spaces of instantons has been the one by Parker and Wolfson [53]. Such a compactification is referred to as the bubble tree compactification.

Another feature indicating the existence of a stable compactification for the instanton moduli space is the fact that in this approach 'punctures never collide'. Therefore, punctures can be considered like fermions, so that there is a sort of underlying exclusion principle. This similarity can be explicitly formulated in the geometrical formulation of quantum Liouville theory [54, 55] and in the framework of anyon theories [56]. The latter corresponds to a problem for particles with configuration space $\overline{\mathcal{M}}_{0,n}$ whose dynamics is described by a quantum hamiltonian which is naturally given by the laplacian with respect to the WP metric (thus giving a self-adjoint operator). In particular, both in Liouville and anyon theory one can associate a conformal weight to elliptic points [55, 56] that in the limit of infinite ramification, corresponding to a puncture, just gives the value 1/2, which is the weight of a fermion. Therefore, in some respect we can consider punctures behaving as noncommutative vertices

$$\{\psi(z_i), \psi(z_j)\} \sim \delta(z_i - z_j)\,. \tag{4.3}$$

This would suggest a possible bridge between the stable compactification and the one considered by Hollowood with the noncommutative U(1) [9]. We also note that in the ADHM construction it should be possible, in principle, to find a suitable embedding of the matrices in $\overline{\mathcal{M}}_{0,n}$ such that the degenerated configurations be naturally compactified à la DKM.

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The emergence of the fermion here has also an interesting feature in connection with the recent developments around Nekrasov's solution and the geometric engineering approach to the $\mathcal{N} = 2$ SYM theory. On one hand, in [57], the τ-conjecture has been proposed, which states that the full-genus (graviphoton corrected) partition function is described by the quantum dynamics of chiral fermions on the SW curve. On the other hand, in [30], it has been discussed that in order to formulate topological vertex in terms of the mirror B-model, which in the end computes relevant amplitudes for SW solution, it is useful to introduce chiral fermion which describes noncompact B-branes.

4.2 The Hurwitz moduli space

Besides instantons, there is another important theory which maps to the moduli space of Riemann surfaces, including the one of punctured spheres. This is the space of meromorphic functions of degree n on genus g Riemann surfaces defining the degree n ramified coverings of the sphere. In the case in which the poles are simple and the critical values of the function sum to 0, this space, denoted by $\mathcal{H}_{g,n}$, is a smooth complex orbifold, fibered over $\mathcal{M}_{g,n}$, whose fiber is naturally defined by associating to each function the corresponding Riemann surface. This space admits a compactification $\overline{\mathcal{H}}_{g,n}$ consisting of stable meromorphic functions [22]. The basic fact is that the projection $\mathcal{H}_{g,n} \longrightarrow \mathcal{M}_{g,n}$ extends to

$$\overline{\mathcal{H}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n} \,. \tag{4.4}$$

A feature of such a projection is that we do not have a vector bundle since the dimension of the fiber may vary by varying the point on the base. One is then interested in the fiberwise projectivization $P\overline{\mathcal{H}}_{g,n}$. As such, this space has a natural two-form given by the first Chern class of the tautological sheaf

$$\psi_{g,n} = c_1(\mathcal{O}(1)) \in H^2(P\overline{\mathcal{H}}_{g,n}) \,. \tag{4.5}$$

A consequence, which is of interest for our purpose, is that the space $P\overline{\mathcal{H}}_{0,n}$ turns out to be fibred on $\overline{\mathcal{M}}_{0,n}$, whose fiber is the projective space PE where

$$E = \oplus_{k=1}^{n} L_k^{\vee} \,, \tag{4.6}$$

is the Whitney sum of the tangent lines to the curve at the punctures. This means that the cohomological algebra is generated by $\psi \equiv \psi_{0,n}$ subject to the relation [58]

$$\psi^n + \sum_{k=1}^{n} \psi^{k-1} c_k(E) = 0 \,. \tag{4.7}$$

It follows that considering the natural projection

$$\pi : P\overline{\mathcal{H}}_{0,n} \longrightarrow \overline{\mathcal{M}}_{0,n} \,, \tag{4.8}$$

we can express any $\alpha \in H^{2d}(P\overline{\mathcal{H}}_{0,n})$ as

$$\alpha = \pi^*(\eta_d) + \pi^*(\eta_{d-1})\psi + \pi^*(\eta_{d-2})\psi^2 + \cdots \,, \tag{4.9}$$

$\eta_k \in H^*(\overline{\mathcal{M}}_{0,n})$. Then, since $\pi_* \psi^s = c_{s-n+1}(-E)$, it follows that the degree of α can be evaluated in terms of integrals on $\overline{\mathcal{M}}_{0,n}$ [59]

$$\deg \alpha = \int_{P\overline{\mathcal{H}}_{0,n}} \frac{\pi^*(\eta_d) + \pi^*(\eta_{d-1})\psi + \pi^*(\eta_{d-2})\psi^2 + \cdots}{1-\psi} = \int_{\overline{\mathcal{M}}_{0,n}} \frac{\eta_d + \eta_{d-1} + \cdots}{c(E)} , \tag{4.10}$$

whose evaluation turns out to be considerable simplified. The above result provides an explicit interesting example that integrals on complicated spaces actually simplify just due to a natural map to $\overline{\mathcal{M}}_{0,n}$. This further supports our program of expressing instanton contributions in terms of integrals on $\overline{\mathcal{M}}_{0,n}$. In this respect, what is of interest is not just the dimensional reduction of the moduli, as it may also happen that there are convenient and higher dimensional parametrizations leading to a simplification, rather we see that $\overline{\mathcal{M}}_{0,n}$ provides a sort of basic space where the complicated integrations simplify considerably.

4.3 The geometry of WP recursion relation

If one looks for a formalism that expresses the $\mathcal{N}=2$ instanton contributions as integrals on the moduli space, in such a way that it respects the bilinear recursive structure of the DKM compactification, one should first understand how the DKM geometry may reflect in the expected bilinear recursion relation. In order to check the existence of such a bilinear recursion relation, we note that a characteristic feature of the classical Liouville theory is that the recursion relation for the WP volumes has particular properties. Let us write down eq. (2.29) once again

$$a_3 = \frac{1}{2} , \qquad a_n = \frac{1}{2}\frac{n(n-2)}{(n-1)(n-3)} \sum_{k=1}^{n-3} a_{k+2} a_{n-k} , \tag{4.11}$$

for $n \geq 4$. The bilinear nature of such a recursion relation is a consequence of the fact that the boundary of the DKM compactification is the union of the products of *two* moduli spaces of lower order. This seems difficult to reproduce in $\mathcal{N}=2$ SYM theories. Actually, it was shown in [50] that a possible analogy was to construct a moduli space for instantons whose boundary has components which also contain the product of *three* subspaces. To discuss this point in some detail we should first better understand the geometrical reason underlying the structure of the recursion relation (4.11). Recalling that it was obtained as the DKM boundary contribution to the integral, we have

$$\dim(\overline{\mathcal{M}}_{0,n}) = \dim(\overline{\mathcal{M}}_{0,k+2}) + \dim(\overline{\mathcal{M}}_{0,n-k}) + 1 , \tag{4.12}$$

$k = 1, \ldots, n-3$, which is satisfied because $\dim(\overline{\mathcal{M}}_{0,n}) = n-3$. It follows that the existence of a formulation in terms of the Liouville F-models, such as expressing the instanton contributions as integrals on the moduli space of punctured spheres, would require that the resulting recursion relation reflects the main properties of the one of WP volumes. Thus, besides being bilinear, it should have other characteristic features which follow from (4.12). In particular the range of the recursion relation should be given by $k = 1, \ldots, n-3$. Furthermore as suggested by (4.12) the indices of the instanton contributions involved in such a, still hypothetical, bilinear recursion relation should be $n-k$ and $k+2$. However, there is a minor subtle point with such an identification. Namely, note that the above data may change by a simple shift of the indices by a global constant and by a shifting of n. More precisely, if we define $a_k = b_{k+m}$ and rewrite eq. (4.11) for $n+m$, we obtain

$$b_{3+m} = 1/2 \,, \qquad b_n = \frac{1}{2}\frac{(n-m)(n-m-2)}{(n-m-1)(n-m-3)} \sum_{k=1}^{n-m-3} b_{k+m+2} b_{n-k} \,, \tag{4.13}$$

for $n \geq 4+m$. This shows that even if it looks different from (4.11), in fact they have the same content. This suggests that in the general bilinear recursion relation

$$d_n = \sum_{k=1}^{p} c_k d_{k+q} d_{n-k} \,, \qquad n \geq r \,, \tag{4.14}$$

one introduces the two quantities that remain invariant under the above transformations, i.e.

$$\mathcal{A} = p + (k+q) + (n-k) = n + p + q \,, \qquad \mathcal{B} = r - q - 1 \,. \tag{4.15}$$

By (4.11), in the case of the WP volumes we find

$$\mathcal{A}_{WP} = 2n - 1 \,, \qquad \mathcal{B}_{WP} = 1 \,. \tag{4.16}$$

5. $\mathcal{N} = 2$ gauge theory as Liouville F-models

In this section we give an explicit construction of instanton amplitudes for $\mathcal{N} = 2$ SU(2) SYM theory on the basis of the Liouville F-models. We first guess the correct dimensionality of the integrand to reproduce the bilinear recursive structure of the amplitudes. Then we discuss the dynamical selection of the boundaries which should be related to the localization of the instanton effects in our formalism. These observations finally lead to our master equation of the $\mathcal{N} = 2$ SYM whose precise form will be determined in the next section by deriving the bilinear recursion relation.

5.1 A master equation in $\mathcal{N} = 2$ SYM?

We have seen that there is a lot of evidence for the existence of a formulation of instanton contributions in terms of integrals on the moduli space of punctured spheres. We should expect that the instanton moduli space can be mapped to $\overline{\mathcal{M}}_{0,n}$ along the lines of what happens in the case of the Hurwitz space. Since the number of parameters for a single instanton does not correspond with the one dimensional complex coordinate of a puncture, we expect that the map should be to the product of moduli spaces of punctured spheres of a particular kind. On dimensional grounds, one should expect that matching of the parameters should associate 4-punctures to each instanton, so we would at first consider the space $\overline{\mathcal{M}}_{0,4n}$ whose dimension is $4n-3$. However, this identification leads to problems with dimensional matching to get the bilinear recursion relation. Actually, this identification will give

$$\overline{\mathcal{M}}_{0,4n} \longrightarrow \overline{\mathcal{M}}_{0,4n-k} \times \overline{\mathcal{M}}_{0,k+2} \,. \tag{5.1}$$

Note that whereas the left hand side refers to a number of punctures which is a multiple of 4, in the right hand side never appear pairs of moduli spaces both having a number of punctures which is multiple of 4. Therefore, with this choice it would be cumbersome to make the identification of, say, the instanton contribution $\mathcal{F}_n$ to the SW prepotential, with an integral over $\overline{\mathcal{M}}_{0,4n}$, and simultaneously satisfying a bilinear recursion relation. However, note that in the case $\overline{\mathcal{M}}_{0,4n+2}$ we would have the pattern

$$\overline{\mathcal{M}}_{0,4n+2} \longrightarrow \overline{\mathcal{M}}_{0,4n-k+2} \times \overline{\mathcal{M}}_{0,k+2} \,, \tag{5.2}$$

so that when k is a multiple of 4 we may identify $\mathcal{F}_n$ with an integral on $\overline{\mathcal{M}}_{0,4n+2}$ and consistently having the bilinear recursion relation.

We then set

$$\mathcal{F}_n = \sum_{k=1}^{n-1} a_k \int_{\mathcal{D}_{4k+2}} \sigma_k^{4k-1} \,, \tag{5.3}$$

where σ_k is some two-form on $\mathcal{D}_{4k+2}$ to be determined. In order to understand the nature of such a form we consider the natural embedding

$$i : \overline{V}^{(4k+2)} \to \overline{V}^{(4k+2)} \times * \to \overline{V}^{(4k+2)} \times \overline{V}^{(4n-4k+2)} \to \partial\overline{V}^{(4n+2)} \to \overline{V}^{(4n+2)} \,, \qquad n > k \,, \tag{5.4}$$

where $*$ is an arbitrary point in $\overline{V}^{(4n-4k+2)}$. On the other hand according to Wolpert theorem we have

$$[\omega_{4k+2}] = i^* \, [\omega_{4n+2}] \,, \qquad n > k \,. \tag{5.5}$$

Using (5.3)(5.4) and (5.5) , we can express $\mathcal{F}_n$ in the form

$$\mathcal{F}_n \sim \int_{\overline{V}^{(4n+2)}} \omega_{4n+2}^{4n-2} \wedge \omega^F \,, \tag{5.6}$$

where $[\omega^F]$ is the dual of a linear combination of the divisors $\mathcal{D}_{4k+2}$. The above investigation suggests that

The instanton contributions can be expressed as integrals on the moduli space of punctured Riemann spheres leading to a bilinear recursion relation.

This allows us to write $\mathcal{F}_n$ in terms of the bilinear recursion of the F-models. Before doing this, we recall that in deriving the master equation we observed that there is a canonical way to select a bilinear recursion relation from the ones which differ by a trivial rescaling of the terms. Therefore, rather than $\mathcal{F}_n$ we will express a rescaled version $\bar{\mathcal{F}}_n$. Note that this is irrelevant as a global rescaling does not change the structure of the recursion relations. We saw that the divisor D^F receives contributions only from the moduli spaces of Riemann spheres with $4k+2$ punctures, that is $F_N(k+2,(N-k))$, $N = 4n+2$, vanishes unless k is multiple of 4. Equivalently, we can consider for D_η the expansion

$$D_\eta = \sum_{k=1}^{n-1} e^{(4k+2)s} e^{(N-4k)t} \langle \omega^F \rangle_{4k+2} \langle \omega^F \rangle_{N-4k} \mathcal{D}_{4k} \,. \tag{5.7}$$

From the master equation, in the case of $\mathcal{N} = 2$ SYM we should then have

$$\bar{\mathcal{F}}_n = \langle \omega^F \rangle_N = F_N(\partial_s, \partial_t) \langle \eta_0 \rangle_N \,, \tag{5.8}$$

which leads to

$$\langle\omega^F\rangle_N = \sum_{k=1}^{n-1} F_N(4k+2,(N-4k))\langle\omega^F\rangle_{4k+2}\langle\omega^F\rangle_{N-4k}\,, \tag{5.9}$$

$n \geq 2$. In the next section we will show that this relation holds and we will fix both $F_N(4k+2,(N-4k))$ and the initial condition $\langle\omega^F\rangle_6$.

5.2 Relation to ADHM construction

We would like to briefly comment on the relation of this proposal to the usual treatment of the integral over moduli spaces of instantons based on the ADHM construction. Explicit instanton calculations show that there appear considerable simplifications in performing integrals over the whole moduli space. This indicates that the original parametrization might not be the most convenient choice in specific cases. The n-th instanton contribution to the prepotential for SU(2) can be obtained from the integration over the ADHM instanton moduli space $\mathfrak{M}_n^I$, whose metric (and hence the volume form $d^{8n}\mu$) is given by (4.2), as

$$\mathcal{F}_n = \frac{1}{\mathrm{Vol}(\mathbb{R}^4)}\int_{\mathfrak{M}_n^I} d^{8n}\mu e^{-S_{\text{eff}}}\,, \tag{5.10}$$

where the effective action S_{eff} is given by the integration over the fermionic coordinates. In this computation, four out of the total $8n$ real coordinates of the moduli space are identified with the center of the instantons. Dividing by the volume of spacetime corresponds to a regularization of the infrared divergence related to the integration over the center. The net result is an integration over the $8n-4$ coordinates left. On the other hand, an interesting result is that this same integration can be rewritten as an integration of the $(4n-3)$-power of a two-form [8]

$$\mathcal{F}_n \simeq \int_{\widehat{\mathfrak{M}}_n^I} (d\rho)^{4n-3}\,. \tag{5.11}$$

Here the moduli space $\widehat{\mathfrak{M}}_n^I$ is obtained from the usual $\mathfrak{M}_n^I$ by first dividing by the center of the instanton as in (5.10) and then by integrating over the complex scale of the ADHM moduli. The closed two-form $d\rho$ can be formally considered as the Euler class associated with the original ADHM moduli space $\mathfrak{M}_n^I$ regarded as a U(1) bundle (see [8] for details). ¿From a dimensional point of view note that both (5.10) and (5.11) have been performed after fixing the center of the instanton coordinates, but the latter gets rid of an additional complex coordinate related to a rescaling of the ADHM moduli. Next we note that it is very useful introducing noncommutative U(1) instantons when regularizing divergences [9], with the effect of smoothing out the singularities of the moduli space. In this case it is convenient to keep the four dimensional explicit integration of the center of the instantons. Therefore it would be natural to consider a moduli space of real dimension $8n-2$, where we simply rescale the moduli. This is just the dimension of $\overline{\mathcal{M}}_{4n+2}$ we selected before.

Let us also note that the DKM boundary of the moduli space of punctured spheres is deeply related to the negatively curved nature of the punctured spheres (with more than 2 punctures). Stability guarantees that spheres with two punctures do not appear. We also note that in the hyperbolic metric the distance between two punctures is infinite. This is in particular the case of the two punctures that remain on removal of the node. However, while the two punctures are infinitely separated in the Poincaré metric, they are at zero distance in the euclidean metric. This is better understood if one considers the upper-half plane: the distance between a point in the upper-half plane and the boundary is divergent as $1/y$ whereas with the euclidean metric we just have y. The crucial remark is that the product of the two distances is a constant. On the other hand, from the euclidean point of view, as we are approaching the boundary of the upper-half space, i.e. a puncture, we are considering an infrared problem, whereas in the hyperbolic metric we are always considering a long-distance, i.e. infrared, geometry. This dual picture is just a sort of UV/IR duality. Actually, this indicates a physical meaning of the DKM compactification, which may lead to a direct connection with the U(1) noncommutative compactification by Hollowood [9]. We also note that the infrared and ultraviolet regularization properties of negatively curved manifolds have been observed in [60]. It is worth also mentioning that such UV/IR regularization is strictly related to the distortion theorems for univalent functions (see the second reference in [2]).

6. The bilinear relation

Until now we have collected evidence that the instanton contributions in $\mathcal{N}=2$ SYM for SU(2) gauge group can be formulated as a deformation of the Liouville theory leading to integrals over the moduli spaces of punctured spheres. Nevertheless, apparently there are several reasons that seem to prevent such a construction. The main obstacle is that the known recursion relation for the instantons is a trilinear one [2]. Let us briefly recall how it was obtained. Consider the relation between the u-modulus and the prepotential [2]

$$u = \pi i \left(\mathcal{F} - a^2 \frac{\partial \mathcal{F}}{\partial a^2} \right) . \tag{6.1}$$

This can be derived either by instanton analysis, [61] therefore without explicitly calculating the $\mathcal{F}_k$ or $\mathcal{G}_k$, or by means of the superconformal anomaly [62]. Taking the derivative of eq. (6.1) with respect to u, one sees that the periods $a_D(u)$ and $a(u)$ satisfy a second-order differential equation without the first derivative term, called the PF equations and whose precise potential is given from the SW curve:

$$\left[\partial_u^2 + \frac{1}{4(u^2-1)} \right] a(u) = 0 . \tag{6.2}$$

If we invert (6.2) for $u = \mathcal{G}(a)$ we find the nonlinear equation

$$(1-\mathcal{G}^2)\mathcal{G}'' + \frac{a}{4}\mathcal{G}'^3 = 0 . \tag{6.3}$$

Expanding in power series $\mathcal{G}(a) = \sum_{k=0}^{\infty} \mathcal{G}_k a^{2-4k}$, one finds the trilinear recursion relation

$$\mathcal{G}_{n+1} = \frac{1}{2(n+1)^2} \left[(2n-1)(4n-1)\mathcal{G}_n + \sum_{k=0}^{n-1} c_{k,n} \mathcal{G}_{n-k} \mathcal{G}_{k+1} - 2 \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} d_{j,k,n} \mathcal{G}_{n-j} \mathcal{G}_{j+1-k} \mathcal{G}_k \right] , \tag{6.4}$$

where $n \geq 0$, $\mathcal{G}_0 = 1/2$ and

$$c_{k,n} = 2k(n-k-1) + n - 1 \qquad d_{j,k,n} = [2(n-j)-1][2n-3j-1+2k(j-k+1)]\,.$$

In this section, we overcome this difficulty and derive the bilinear recursion relation. As a result we determine $F_N(4k+2,(N-4k))$ and the initial condition $\langle \omega^F \rangle_6$ for the master equation of the $\mathcal{N}=2$ SYM theory.

6.1 Inverting differential equations

The trilinear recursion relation (6.4) is unsatisfactory for our purpose because it does not respect the essential bilinear recursive property of the DKM compactification with which our master equation naturally is equipped. Therefore it is natural to ask a question when we can obtain the bilinear recursion relation starting from the general potential $V(x)$ in the PF equation. In this subsection and the next, we derive the necessary conditions to obtain a bilinear recursion relation in this setup.

Since the following construction is general, we will use $\psi(x)$ rather than $a(u)$. Let us consider the second-order differential equation

$$\left[\partial_x^2 + V(x)\right]\psi(x) = 0\,, \tag{6.5}$$

and set $x = \mathcal{G}(\psi)$. Since $\partial_x = \mathcal{G}'^{-1}\partial_\psi$ and $\partial_x^2 = -\mathcal{G}'^{-3}\mathcal{G}''\partial_\psi + \mathcal{G}'^{-2}\partial_\psi^2$, where here $' \equiv \partial_\psi$, it follows that equation for $\mathcal{G}(\psi)$ satisfies the differential equation

$$V^{-1}\mathcal{G}'' = \psi\mathcal{G}'^3\,. \tag{6.6}$$

Now suppose we want to expand $\mathcal{G}(\psi)$ in power series and gain a recursion relation for the coefficients of this expansion from (6.6). Clearly the relation we will obtain would be trilinear at least, due to the appearance of the third power of $\mathcal{G}'$. Fortunately enough, it turns out that for a certain class of functions $V(x)$, there is a nice way to simplify this nonlinear equation in order to obtain a bilinear equation. This holds only in the case where $V^{-1}(x)$ is at most quadratic in its argument.

6.2 From trilinear to bilinear

Introduce a function $\mathcal{H}(\psi)$ such that $\mathcal{H}' = \mathcal{G}(\psi)$ and define the auxiliary function f through the following relation

$$V^{-1} + f\mathcal{H}'' = 0\,, \tag{6.7}$$

so that a factor $\mathcal{H}''$ drops and (6.6) becomes

$$f\mathcal{H}''' + \psi\mathcal{H}''^2 = 0\,. \tag{6.8}$$

Differentiating (6.7) we have by (6.8)

$$f'\mathcal{H}'' - \psi\mathcal{H}''^2 + \partial_\psi V^{-1} = 0\,. \tag{6.9}$$

On the other hand, since V is a function of $\mathcal{G} = \mathcal{H}'$, we have $\partial_\psi V^{-1} = \mathcal{H}''\partial_{\mathcal{H}'}V^{-1}$, so that (6.9) becomes

$$f' - \psi\mathcal{H}'' + \partial_{\mathcal{H}'}V^{-1} = 0\,, \tag{6.10}$$

which we can easily integrate to obtain the following expression for the auxiliary function

$$f = \psi \mathcal{H}' - \mathcal{H} - \int^{\psi} d\tilde{\psi} \partial_{\mathcal{H}'} V^{-1} \,. \tag{6.11}$$

Plugging this solution into the defining equation (6.7) for f, we obtain our final equation

$$\left(\psi \mathcal{H}' - \mathcal{H} - \int^{\psi} d\tilde{\psi} \partial_{\mathcal{H}'} V^{-1} \right) \mathcal{H}'' + V^{-1} = 0 \,. \tag{6.12}$$

We then have

Proposition. *By (6.12) it follows that if $V^{-1}(x)$ is a polynomial at most quadratic in x; then the recursion relation for the coefficients of the power expansion of $x = \mathcal{G}(\psi)$, with $\psi(x)$ solution of (6.5), is bilinear.*

6.3 The $\mathcal{N} = 2$ bilinear relation

We now readily apply this procedure to the PF equation (6.2). By introducing $\mathcal{H}'(a) = \mathcal{G}(a)$ and following the above steps we find the following nonlinear equation

$$1 - \mathcal{H}'^2 = \frac{1}{4} \mathcal{H}'' (a \mathcal{H}' - 9 \mathcal{H}) \,. \tag{6.13}$$

Note the while (6.3) was trilinear, this last (6.13) is just bilinear. Plugging the asymptotic expansion

$$\mathcal{H}(a) = \sum_{k=0}^{\infty} \frac{\mathcal{G}_k}{3 - 4k} a^{3-4k} \,, \tag{6.14}$$

into (6.13) we eventually find the bilinear recursion relation for $\hat{\mathcal{G}}_n \equiv \mathcal{G}_n / (4n - 3)$

$$\hat{\mathcal{G}}_n = \sum_{k=1}^{n-1} g_{k,n} \hat{\mathcal{G}}_k \hat{\mathcal{G}}_{n-k} \,, \tag{6.15}$$

where $n \geq 2$, $\hat{\mathcal{G}}_1 = 1/2^2$ and

$$g_{k,n} = \frac{(2n - 15)k(n - k)}{n^2} + 3 \,. \tag{6.16}$$

This shows in $\mathcal{N} = 2$ SYM we have the master equation (5.8), where we identify $\bar{\mathcal{F}}_n = \hat{\mathcal{G}}_n$ and

$$F_N = 3 + \frac{2n - 15}{16n^2} (\partial_s - 2)(\partial_t - 2) \,, \qquad \langle \omega^F \rangle_6 = \frac{1}{2^2} \,. \tag{6.17}$$

By equation (6.1) we obtain the relation between the coefficients of the prepotential $\mathcal{F}(a) = \sum_k a^{2-4k} \mathcal{F}_k$ and $\mathcal{G}_k$, namely $\mathcal{G}_k = 2\pi i k \mathcal{F}_k$, $k \geq 1$. Then we obtain the bilinear recursion relation for the instanton expansion of the prepotential as

$$\frac{\mathcal{F}_n}{2\pi i} = \frac{4n - 3}{n} \sum_{k=1}^{n-1} e_{k,n} \mathcal{F}_k \mathcal{F}_{n-k} \,, \tag{6.18}$$

where

$$e_{k,n} = \frac{k(n-k)}{(4k-3)[4(n-k)-3]} g_{k,n} \,. \tag{6.19}$$

This completes our construction of the instanton moduli space in terms of the moduli space of punctured spheres and the consequent derivation of the master equation for $\mathcal{N} = 2$ SYM theory proposed in section 5. Also note that we have the same invariants introduced in section 4.3:

$$\mathcal{A}_{WP} = \mathcal{A}_{\mathcal{N}=2} \,, \tag{6.20}$$

$$\mathcal{B}_{WP} = \mathcal{B}_{\mathcal{N}=2} \,. \tag{6.21}$$

As a final remark, we note that the auxiliary function $\mathcal{H}' = \mathcal{G}$ can be actually integrated by the PF equation (6.2) to

$$\mathcal{H} = \frac{u}{9} a + \frac{4}{9}(u^2 - 1) a' \,. \tag{6.22}$$

Observe that $\mathcal{H}$ has the same monodromy of the period a. This suggests the possibility to introduce its dual

$$\mathcal{H}_D = \frac{u}{9} a_D + \frac{4}{9}(u^2 - 1) a'_D \,, \tag{6.23}$$

satisfying the equation $\partial_{a_D} \mathcal{H}_D = u$ which is the dual of $\mathcal{H}' = u$. Also note that by $\partial_{a_D} \mathcal{H}_D = \partial_a \mathcal{H}$ we have

$$\frac{\partial \mathcal{H}_D}{\partial \mathcal{H}} = \tau \,. \tag{6.24}$$

Since $\mathcal{H}$ naturally appears in deriving the bilinear relation, it should be further considered in the framework of SW theory. Its physical interpretation, which may appear in the strong coupling region, considered in [63], needs to be uncovered.

6.4 The SW solution

Before going to the stringy interpretation of our result, we would like to point out an interesting possibility. Our discussion of the derivation of the bilinear recursion relation is based on the known SW solution. However, if we turn the logic around and first assume the existence of the bilinear recursion relation *without* specifying its coefficients, we can *derive* the SW solution. In other words, if we know the instanton amplitude is given by a certain (unknown) Liouville F-model beforehand, we can reproduce the SW solution, and hense specify the actual Liouville F-model. Our assumption is

> *The instanton contributions can be expressed as integrals on the moduli space of punctured Riemann spheres leading to a bilinear recursion relation.*

We will also make use of the one-instanton contribution, which is $\mathcal{G}_1 = 1/2^2$, and of the relation between the u-modulus and the prepotential [2]

$$u = \pi i \left(\mathcal{F} - a^2 \frac{\partial \mathcal{F}}{\partial a^2} \right) . \tag{6.25}$$

This relation can be derived either by instanton analysis, that is proving $\mathcal{G}_k = 2\pi i k \mathcal{F}_k$ directly, [61, 64] therefore without explicitly calculating the $\mathcal{F}_k$ or $\mathcal{G}_k$, or by means of the superconformal anomaly [62].

From the relation (6.25), we know that $a(u)$ satisfies the PF equation with an unknown potential $V(u)$. However, according to the Proposition, V^{-1} should have the form

$$V^{-1}(u) = Au^2 + Bu + C\,. \tag{6.26}$$

Recalling that $u = \mathcal{G}(a) = \mathcal{H}'(a)$, we have by (6.12)

$$\mathcal{H}''[a\mathcal{H}' - (1+2A)\mathcal{H} - Ba] + A\mathcal{H}'^2 + B\mathcal{H}' + C = 0\,. \tag{6.27}$$

Putting the asymptotic expansion

$$\mathcal{H}(a) = \sum_{k=0}^{\infty} \frac{\mathcal{G}_k}{3-4k} a^{3-4k}\,, \tag{6.28}$$

with $\mathcal{G}_0 = 1/2$, which follows by the asymptotic expansion of $\mathcal{F}(a)$ and eq. (6.25), we obtain

$$\sum_{j,k\geq 0} \mathcal{G}_j \mathcal{G}_k a^{4-4(j+k)} \left[4\frac{(2k-1)(2j+A-1)}{3-4j} + A\right] + B\sum_{j\geq 0} \mathcal{G}_j (4j-1) a^{2-4j} + C = 0\,. \tag{6.29}$$

Now observe that each term a^{2-4k} is multiplied only by a singular $\mathcal{G}_k$, so that

$$B = 0\,. \tag{6.30}$$

Next, the term a^2 gives

$$A = 4\,. \tag{6.31}$$

The coefficient C is determined by requiring that $\mathcal{G}_1 = 1/2^2$, which gives

$$C = -4\,. \tag{6.32}$$

Therefore we have obtained the PF equation for $\mathcal{N}=2$ SYM for SU(2) gauge group

$$\left[\partial_u^2 + \frac{1}{4(u^2-1)}\right] a(u) = 0\,. \tag{6.33}$$

We have then shown that the conjectured relation between $\mathcal{N}=2$ SYM and the Liouville theory leads us to an assumption that the bilinear recursion relation for $\mathcal{N}=2$ exists, and this assumption together with the 'initial condition', which in the case of $\mathcal{N}=2$ is given by the values of $\mathcal{G}_0$ and $\mathcal{G}_1$, is sufficient to completely fix the whole solution.

7. Geometric engineering and noncritical string theory

In this section we show that the map from the instanton moduli space to the moduli space of punctured Riemann surfaces have a counterpart in terms of the geometric engineering approach and the noncritical string theory. These approaches shed new light on the physical origins of this map. The similarity of the derivation suggests that our recursion relation strictly reflects the duality among the different descriptions.

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7.1 Geometric engineering

We have seen that we can map the problem of calculating instanton contributions to integrals on the moduli space of the punctured spheres. We now discuss how this may be interpreted in terms of the geometric engineering formulation. Consider the bilinear recursion relation as in (6.18), which has the same structure we encounter in the Liouville F-models. In particular, a nice feature of the matching of the dimensions in (4.12) is that it holds for a generic decomposition of the moduli space of a $(r+2)$-punctured sphere, due to its DKM compactification origin. In particular, we can now identify the n-instanton coefficient with an integral over the moduli space of an n-punctured sphere. Let us consider the Liouville F-models and set $Z_n^F = \mathcal{F}_{n-2}$, then the recursion relation of Z_n^F (3.19) is translated as

$$\mathcal{F}_n = Z_{n+2}^F = \sum_{k=1}^{n-1} F_{n+2}(k+2, n+2-k)\mathcal{F}_k\mathcal{F}_{n-k}\,. \tag{7.1}$$

Now if we set

$$\frac{1}{2\pi i}F_{n+2}(k+2, n+2-k) = \frac{4n-3}{n}e_{k,n}\,, \tag{7.2}$$

we obtain the desired recursion relation (6.18). Note that the divisor introduced here consists of all the possible number of punctures, as opposed to that of the previous section, where the divisor or its coefficients $F(n,k)$ vanish unless k is multiple of 4. In the reduction here, we no longer need to restrict our boundaries to the multiple of 4, which we believe is a more compact formulation.[14] Though we have lost the perspective as the integration over the instanton moduli space, we have a new interpretation of this integration over the moduli space of the n-punctured spheres as the perturbative calculation of the free-energy in the topological string theory. Furthermore it yields a clue to extend our formulation to the case in which the gauge theory is coupled to a gravitational background (see e.g. [65, 66] for a comparison of the geometric engineering expression for the graviphoton corrected prepotential with Nekrasov's formula). The argument goes as follows.

First, let us recall the geometric engineering approach [33] to obtain the Seiberg-Witten prepotential. In this approach, we consider a topological A-model on a certain local Calabi-Yau space and the free-energy of this theory yields the whole prepotential of the $\mathcal{N}=2$ SYM theory. In our SU(2) case, we take the local Hirzebruch surface ($\mathbb{P}^1$ bundle over $\mathbb{P}^1$). Local here means that we consider the canonical line bundle over the Hirzebruch surface to construct a noncompact Calabi-Yau threefold. The proper identification of the parameters between the gauge theory and the geometry is

$$e^{-T_B} = \left(\frac{\beta\Lambda}{2}\right)^4, \qquad T_F = 2a\beta\,, \qquad g_s = \beta\hbar\,, \tag{7.3}$$

where T_B is the volume of the base $\mathbb{P}^1$, T_F is that of the fiber and the parameter β is introduced to obtain the four dimensional field theory limit as $\beta \to 0$. This setup can be obtained also by considering M-theory on the same Calabi-Yau threefold times a circle

[14] In order to obtain the canonical form introduced in section 3, we need to rescale $\mathcal{F}_n$ and use $\hat{\mathcal{G}}_n$ instead. Setting $Z_n^F = \hat{\mathcal{G}}_{n-2}$, we obtain $F_{n+2}(k+2, n+2-k) = g_{k,n}$.

of radius $\beta/2\pi$, obtaining a five dimensional gauge theory with eight supercharges [67]. The four dimensional field theory limit $\beta \to 0$ here corresponds to shrinking the circle to zero size. Furthermore, the string coupling $\hbar$ in the A-model is identified with the four dimensional self-dual graviphoton field strength F_+ [68, 69] and, after taking the field theory limit, this allows us to identify the higher genus topological string amplitudes with the gravitational corrections to the prepotential $\mathcal{F}(a,\hbar)$ of the $\mathcal{N}=2$ four dimensional gauge theory [57, 66].[15] Therefore, if we restrict ourselves to the usual gauge theory on flat background, what we need to evaluate is just the genus zero free-energy of the topological CFT. Particularly the n-th gauge theory instanton contribution comes from worldsheet instantons that wrap n times the base $\mathbb{P}^1$.

To connect this fact with our expression of the prepotential, which is described by the integration over the moduli space of the n-punctured spheres, we will consider the A-model from a world sheet point of view. Consider the period a of the gauge theory and take the $a \to \infty$ limit of the above setup, which corresponds to the semiclassical limit in the field theory, thus obtaining an A-model CFT whose action we denote as S_∞. Then for finite a the world sheet action can be written as a perturbed CFT with respect to the original S_∞

$$S_a = S_\infty + \frac{1}{a^4}\int d^2z O(z)\,, \tag{7.4}$$

for a certain operator $O(z)$. The $1/a^4$ expansion of the SW prepotential $\mathcal{F}(a)$ will be given by the free-energy of this latter CFT, which we can evaluate as a series in the perturbation.[16] Usually, the A-model free-energy is computed by the world sheet instanton summation (Gromov-Witten invariants), but here we would like to regard the A-model just as an abstract topological CFT instead.[17] Now suppose we want to evaluate the free-energy of this CFT: the perturbative insertion of the vertex operator $\frac{1}{a^4}\int d^2zO(z)$ will give us an extra moduli (puncture) and the integration over its position. Thus this nicely reproduces our expression of the prepotential as the integration over the moduli space of the n-punctured spheres. Therefore, on the basis of our Liouville F-model analysis, we expect that

$$\mathcal{F}_n \sim \int_{\overline{\mathcal{M}}_{0,n+2}} {\omega_{0,n+2}}^{n-2} \wedge \omega^I \sim \left\langle \left(\int d^2zO(z)\right)^n \right\rangle_{S_\infty}\,. \tag{7.5}$$

The merit of this interpretation is that we know how to incorporate the graviphoton correction in the A-model side. It is essentially given by the higher genus free-energy as explained in (7.3) [68, 69]. This provides a clue to extend our formulation to the gravitational background: we simply consider the same CFT on higher genus Riemann surfaces. It would be very interesting to complete this program and we leave it as a future

[15] The difference among the various kind of Hirzebruch surfaces is irrelevant in this limit.

[16] This is only true if $1/a^4$ is a special coordinate of the topological CFT. Though it is a difficult problem to verify this, our formulation suggests that this is indeed true. We also expect that $O(z)$ is formally BRST exact so that it contributes to the amplitude only through the contact terms, as we can see from our recursion relation. The whole construction seems more transparent if we use the mirror symmetry and move to the B-model, but in the abstract CFT language used here, there is no essential difference.

[17] We postpone the connection with the Gromov-Witten invariants to the last section.

study. It should be noted, however, that though we expect that the graviphoton corrected prepotential is given by the integration over the moduli space of the punctured higher genus Riemann surfaces, the integrand needs not a priori coincide with (7.5).

7.2 Noncritical string

Finally, we present a noncritical string interpretation of the bilinear recursion relation. As we have stated in the introduction, the noncritical string theory and the supersymmetric gauge theory have many common features. There we have discussed the relation to the world sheet U(1) instanton. However, when we consider the space-time theory, things become more interesting. We have learned that the instanton part of the $d = 4$, $\mathcal{N} = 2$ SYM coupling constant $\tau = \frac{1}{2\pi i}\sum_k \tau_k a^{-4k}$ obeys by (6.18) the bilinear recursion relation

$$\tau_n = \sum_{k=1}^{n-1} e'_{k,n}\tau_k\tau_{n-k}\,, \tag{7.6}$$

where

$$e'_{k,n} = \frac{(4n-3)(4n-2)(4n-1)}{n[4(n-k)-1][4(n-k)-2](4k-1)(4k-2)}e_{k,n}\,. \tag{7.7}$$

The reason why we consider τ instead of $\mathcal{F}$ will be clarified later by the strong analogy to the $c = 0$ Liouville theory where we mainly study the specific heat, but here we only note that it is dimensionless. We would like to understand this relation in terms of the string genus expansion.

It was proposed in [42] that this kind of recursion relation can be rewritten as the perturbative noncritical string form (there it was done for the Painlevé I — the $c = 0$ noncritical string theory). Rewriting in this form enabled us to understand the origin of reduction of the path integration over the Liouville field into the known cohomological objects of the moduli space of Riemann surface (see also the recent argument of Zamolodchikov [70] for a possible explanation of this reduction mechanism).

Here, rewriting the SYM recursion relation in terms of a string perturbation theory touches some interesting issues. First, we might find a confining string theory behind SYM, which we name as instanton string theory. Second, we may have a better understanding on the negative expansion of the cosmological constant in the Liouville theory.[18] It should also be noted that the proposal that the noncritical string theory is deeply connected with the $4d$ (supersymmetric) gauge theory can be found in Polyakov [71] before the discovery of AdS/CFT. He considered a certain noncritical string theory as a dual description of the YM theory. The usual difficulty is that the noncritical string theory has a $c = 1$ barrier, but he proposed that in the higher dimension, the Liouville dimension becomes the warped geometry instead. In our opinion, this proposal has a nice realization in the $\mathcal{N} = 2$ super Liouville theory: thanks to the discovery of the duality in the $\mathcal{N} = 2$ Liouville theory conjectured in [72] and proved in [24] (see also [29, 73] for a dual picture and a relation

[18] It is impossible to understand this in terms of the world sheet instanton theory. The negative dependence of μ cannot be understood from the usual instanton expansion. This is also true in the usual Goulian-Li approach of the Liouville theory in the higher genus.

to the SW theory) the Liouville potential has a dual realization as the warped geometry.[19] The duality between the B-model on conifold and the $c = 1$ noncritical string theory [74] is another example of this phenomenon. The partition function of the conifold describes the universal nonperturbative effect of the $\mathcal{N} = 1$ supersymmetric gauge theories.

In our $\mathcal{N} = 2$ SYM theory, the genus expansion is given by $\left(\frac{\Lambda}{a}\right)^{4g}$ in the SYM side. On the noncritical string side, we expect $g_s^{2g}\mu_r{}^{-2g}$, where μ_r is the renormalized cosmological constant. Thus the correspondence is

$$\Lambda^4 = \Lambda_0^4 e^{-4\left(\frac{8\pi^2}{g^2} - i\theta\right)/b} \sim g_s^2 = e^{2\langle\phi\rangle} , \qquad a^4 \sim \mu_r^2 , \tag{7.8}$$

where $b = 2N_c - N_f = 4$ in our particular SU(2) case. Looking at this expression, we make two observations. On the one hand, the dependence of the θ parameter in the SYM side reminds us of the stringy Θ parameter (3.27) proposed by Bonelli-Marchetti-Matone [18]. Secondly, the correspondence between a and μ_r may have the geometrical origin. In the $\mathcal{N} = 2$ Liouville - CY correspondence, μ_r plays the rôle of the deformation of the complex moduli parameter, and it should be related to the moduli of the $4d$ theory — in this case, of course, the $\mathcal{N} = 2$ SYM moduli a.

Furthermore, it is interesting to see the connection between the KPZ scaling law of the noncritical string theory and the (fractional) instanton contribution of the supersymmetric gauge theory to the physical quantities. In the KPZ scaling law, the path integration over the Liouville zero mode leads us to the genus scaling law of the cosmological constant which we identify with the SYM moduli. On the other hand, in the $\mathcal{N} = 2$ pure SYM theory, nonperturbative corrections to the prepotential come only from the instanton and not from fractional instantons in contrast to the $\mathcal{N} = 1$ gauge theories. These structures in the gauge theories are also determined from symmetry arguments as is the case with the Liouville theory. This coincidence should have the same origin as the emergence of the $\mathcal{N} = 2$ Liouville theory as the world sheet theory of the corresponding gauge theories. In this perspective, the fact that the amplitudes involving fractional instantons, which is typical in $\mathcal{N} = 1$ gauge theories, usually demand an analytic continuation corresponds to the fractional power of the cosmological constant in the Liouville theory, which also demands an analytic continuation procedure to evaluate correlators.

Now, in order to obtain the noncritical string expression for the instanton string theory, we will repeat the argument given in [42] in our context. We consider moduli spaces of higher genus punctured Riemann surfaces and define the expectation value in the corresponding Liouville background as[20]

$$\langle\sigma\rangle_{g,n} \equiv \frac{1}{(3g-3+n)!} \int_{\overline{\mathcal{M}}_{g,n}} \omega_{g,n}{}^{3g-2+n} \wedge \sigma , \tag{7.9}$$

where we follow the notations introduced in section 3. This Liouville background terminology makes sense here because S_{cl} is the potential for the WP metric even for the punctured

[19] One minor difference, however, is that the warped coordinate is only the Liouville direction.

[20] The omitted suffix to σ is the same of the one appearing as suffix of the bracket. We will use this notation throughout this subsection.

higher genus Riemann surfaces (see [35]). We define the genus expansion of the $\mathcal{N}=2$ effective coupling constant as[21]

$$\tau_g = \langle \omega_I \rangle_{g,2} \,. \tag{7.10}$$

Let us introduce the basis of divisors $D_0 = \overline{\mathcal{M}}_{g-1,4}$, $D_k = \overline{\mathcal{M}}_{k,2} \times \overline{\mathcal{M}}_{g-k,2}$, $k = 1, \ldots, g-1$. Furthermore it is convenient to rescale the divisors by the normalized WP volumes as we have done in section 3 so that $(\langle \omega \rangle_{i,j} \equiv \langle \omega_{i,j} \rangle_{i,j})$

$$\mathcal{D}_0 = \frac{D_0}{\langle \omega \rangle_{g-1,4}} \,, \qquad \mathcal{D}_k = \frac{D_k}{\langle \omega \rangle_{k,2}} \,. \tag{7.11}$$

We now define a divisor D_I, which we call '$\mathcal{N}=2$ divisor', as the $(6g-4)$-cycle

$$D_I = c_0^{(g)} \langle \omega_I \rangle_{k,4} \mathcal{D}_0 + \sum_{k=1}^{g-1} c_k^{(g)} \langle \omega_I \rangle_{k,2} \langle \omega_I \rangle_{g-k,2} \mathcal{D}_k \,, \tag{7.12}$$

where the coefficients $c_k^{(g)}$ will be given later. We identify $[\omega_I]$ as the Poincaré dual to D_I, i.e. $[\omega_I] = c_1([D_I])$ where, as usual, $[D]$ denotes the line bundle associated to a given divisor D and c_1 is the first Chern class. Our rescaling of the divisor is based on the same spirit in section 3.

We now fix the $c_k^{(g)}$'s by requiring that τ_g's defined in (7.10) satisfy the recursion relation (7.6). Two facts are crucial to obtain recursion relation: first, in evaluating the relevant integrals will appear only the components of the boundary $\partial\overline{\mathcal{M}}_{g,2}$ of the form $\overline{\mathcal{M}}_{g-k,i} \times \overline{\mathcal{M}}_{k,j}$ with $i=j=2$ and $\overline{\mathcal{M}}_{g-1,4}$, second, $\omega_{g,2}$ satisfies the restriction phenomenon mentioned above. In particular, considering the natural embedding

$$i: \ \overline{\mathcal{M}}_{k,2} \rightarrow \overline{\mathcal{M}}_{k,2} \times * \rightarrow \overline{\mathcal{M}}_{k,2} \times \overline{\mathcal{M}}_{g-k,2} \rightarrow \partial\overline{\mathcal{M}}_{g,2} \rightarrow \overline{\mathcal{M}}_{g,2} \,, \tag{7.13}$$

$g > k$, where $*$ is an arbitrary point on $\overline{\mathcal{M}}_{g-k,2}$, one has by the Wolpert theorem [39]

$$[\omega_{k,2}] = i^* [\omega_{g,2}] \,, \tag{7.14}$$

and similarly for $\omega_{g-1,4}$. By Poincaré duality one obtains

$$\langle \omega_I \rangle_{g,2} = \frac{1}{(3g-1)!} [\omega_{g,2}]^{3g-2} \cap [D_I] \,, \tag{7.15}$$

so that by (7.13) and (7.14) and setting

$$c_0^{(g+1)} = 0 \,, \tag{7.16}$$

we have

$$\tau_g = \frac{1}{(3g-1)!} \sum_{k=1}^{g-1} c_k^{(g)} \langle \omega_I \rangle_{k,2} \langle \omega_I \rangle_{g-k,2} [\omega_{k,2} + \omega_{g-k,2}]^{3g-2} \cap \mathcal{D}_k \,. \tag{7.17}$$

[21] Since prepotential is naturally connected to the free-energy, we use its second derivative, that is, the coupling constant here to compare the result with [42]. The emergence of $\overline{\mathcal{M}}_{g,2}$ is naturally understood in this context. This moduli space is related to the topological nature of the recursion relation.

The only contribution in the r.h.s. comes from (recall $\mathcal{D}_k$ is proportional to $\overline{\mathcal{M}}_{k,2} \times \overline{\mathcal{M}}_{g-k,2}$)

$$\omega_{g-k,2}{}^{3(g-1-k)+2} \wedge \omega_{k,2}{}^{3k-1} , \tag{7.18}$$

therefore we have

$$\tau_g = \frac{1}{3g-1} \sum_{k=1}^{g-1} c_k^{(g)} \tau_{k,2} \tau_{g-k,2} , \tag{7.19}$$

The recursion relation (7.19) coincides with (7.6) if we set

$$c_k^{(g)} = (3g-1) e'_{k,g} , \tag{7.20}$$

for $k > 0$. Notice that all the coefficients $c_k^{(g)}$ are rational numbers so that D_I defines a rational homology class and the above computations can be interpreted in the sense of rational intersection theory.

Substituting (7.19) into $\tau(a) = \frac{1}{2\pi i} \sum_g \tau_g a^{-4g}$, we have asymptotically

$$\tau(a) \sim \text{perturbative part} + \frac{1}{2\pi i} \sum_{g=1}^{\infty} a^{-4g} \langle \omega_I \rangle_{g,2} . \tag{7.21}$$

So far we have obtained the instanton coefficients as formal integration of cohomological objects over the moduli space of the Riemann surface. If we want to identify this as a string perturbation theory, we should address the question of finding a worldsheet action for the noncritical string theory. In this respect note that this structure arises as Liouville F-models. In our context, the difference is only the coefficient of the divisor. Therefore we expect that the instanton string theory for $\mathcal{N} = 2$ SYM is given by a certain noncritical string theory though we cannot give a precise action at this moment. This perspective will eventually connect the $\mathcal{N} = 2$ SYM theory with a noncritical string theory as was proposed by Polyakov. Furthermore, in this view, we can naturally understand the origin of the bilinear recursion relation as the string equation of the instanton string theory.

For a preliminary study of this connection, we propose a possible world sheet reduction mechanism of the recursion relation as has been done in the case of $c = 0$ Liouville theory. For this purpose we give a conjectural argument that could relate our *ansatz* (7.10) to the path-integral approach to our instanton string theory. In this line of thought a key step is the Duistermaat-Heckman (DH) theorem [75] which roughly speaking corresponds to the following statement.[22] Let X be a $2n$-dimensional symplectic manifold with symplectic form ω and H a hamiltonian on X. Then integrals such as

$$\frac{1}{n!} \int_X \omega^n e^{-H} , \tag{7.22}$$

only depend on the behavior of the integrand near the critical points of the flow of the hamiltonian vector field. The point is that in a path-integral approach one expects that

[22]Duistermaat-Heckman theorem is also essential in the instanton calculation. See e.g. [5] and references therein. Thus if we study the connection with our formalism here, it will lead to some information on the reduction mechanism in the noncritical string theory.

the contribution at genus g to the two-puncture correlation function of instanton string theory which we identify as the coupling constant (because it is the second derivative of the free-energy) is given by[23]

$$\langle {\mathcal{O}_0}^2\rangle_g^{\rm CFT} = \langle \omega\, e^{-H}\rangle_{g,2} \equiv \frac{1}{(3g-1)!}\int_{\overline{\mathcal{M}}_{g,2}} {\omega_{g,2}}^{3g-1} e^{-H}\,, \tag{7.23}$$

where H is an 'effective action' arising from the integration in the path-integral at fixed moduli in $\overline{\mathcal{M}}_{g,2}$. The two-form $\omega_{g,2}$ is symplectic on $\overline{\mathcal{M}}_{g,2}$, regular in the interior and extending as a current to the boundary, therefore, regarded as a map from $T^*\overline{\mathcal{M}}_{g,2}$ to $T\overline{\mathcal{M}}_{g,2}$, ${\omega_{g,2}}^{-1}$ has zeroes only on $\partial\overline{\mathcal{M}}_{g,2}$. Furthermore, since $\omega_{g,2}$ is a Kähler form, the hamiltonian vector field is given by ${\omega_{g,2}}^{-1}dH$ so that the flow of the hamiltonian vector field has critical points at $\partial\overline{\mathcal{M}}_{g,2}$. Let us assume that DH applies to the integral (7.23) and furthermore it gets contributions only from the critical points in the component of $\partial\overline{\mathcal{M}}_{g,2}$ whose factor contain an even number of punctures. Then one expects

$$\begin{aligned}\langle {\mathcal{O}_0}^2\rangle_g^{\rm CFT} &\sim \alpha_g\langle\omega\, e^{-H}\rangle_{g-1,4} + \sum_{k=1}^{g}\beta_{g,k}\langle\omega\, e^{-H}\rangle_{k,2}\langle\omega\, e^{-H}\rangle_{g-k,2}\\ &= \alpha_g\langle {\mathcal{O}_0}^4\rangle_{g-1}^{\rm CFT} + \sum_{k=1}^{g-1}\beta_{g,k}\langle {\mathcal{O}_0}^2\rangle_k^{\rm CFT}\langle {\mathcal{O}_0}^2\rangle_{g-k}^{\rm CFT}\,,\end{aligned} \tag{7.24}$$

where α_g, $\beta_{g,k}$ are possibly a-dependent coefficients. The difference from the $c=0$ Liouville theory studied in [42] is that β has a k dependence. Let us introduce the cohomology classes $[\sigma_{g,k}] \in H^2\left(\overline{\mathcal{M}}_{g,2}\right)$, $k=0,\ldots,g-1$, Poincaré dual of $\mathcal{D}_k$ defined in (7.11). Introducing the normalized WP volumes $\langle\omega_{g,n}\rangle_{g,n}$ and using the fact that $\langle {\mathcal{O}_0}^4\rangle^{\rm CFT}$ is related to the second derivative of $\langle {\mathcal{O}_0}^2\rangle^{\rm CFT}$, one obtains for the asymptotic behavior of the correlation functions

$$\langle {\mathcal{O}_0}^2\rangle_g^{\rm CFT} \sim \tilde\alpha_g\langle {\mathcal{O}_0}^2\rangle_{g-1}^{\rm CFT}\langle\sigma_{g,0}\rangle_{g,2} + \sum_{k=1}^{g-1}\beta_{g,k}\langle \mathcal{O}_0^2\rangle_k^{\rm CFT}\langle {\mathcal{O}_0}^2\rangle_{g-k}^{\rm CFT}\langle\sigma_{g,k}\rangle_{g,2}\,, \tag{7.25}$$

where $\tilde\alpha_g = \alpha_g\frac{25(g-1)^2-1}{4}$.

Since the asymptotic expression of $\langle {\mathcal{O}_0}^2\rangle_g^{\rm CFT}$ evaluated at $a=1$ is equal to $\frac{1}{2\pi i}\tau_g$, by setting $a=1$ in (7.24) we derive

$$\tau_g \sim \langle\omega^{(g,2)}\rangle_{g,2}\,, \tag{7.26}$$

with $\omega^{(g,2)}$ a two-form given by

$$\omega^{(g,2)} = \tilde\alpha_g\langle\omega^{(g-1,2)}\rangle_{g-1,2}\sigma_{g,0} - \sum_{k=1}^{g-1}\beta_{g,k}\langle\omega^{(k,2)}\rangle_{k,2}\langle\omega^{(g-k,2)}\rangle_{g-k,2}\sigma_{g,k}\,, \tag{7.27}$$

exhibiting the same structure of the two-form ω_I introduced in the *ansatz* (7.10) with the coefficients $c_k^{(g)}$ given by (7.20).

Particularly, note that $\tilde\alpha_g = 0$. This should have an interesting physical origin. Moreover this construction clearly explains the natural emergence of the normalization by the WP volume factor in the recursion relation from the localization technique.

[23]In this subsection we use the notation $\langle\,\rangle^{\rm CFT}$ when referring to CFT correlators as opposed to the usual $\langle\,\rangle$ notation for the Liouville background.

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8. Discussion

So far, we have obtained the three different expressions for the prepotential (or the effective gauge coupling constant), starting from our bilinear relation: the integration over the moduli space of the $(4n+2)$-punctured spheres, the integration over the moduli space of the n-punctured spheres, and the genus expansion ('instanton string theory'). These expressions are physically interpreted as the direct instanton calculation, the topological A-model expansion, and the noncritical string theory respectively. We emphasize that the three different approaches to the SW theory is deeply connected via our bilinear recursion relation.

Here we would like to discuss the nature of the duality among these expressions from our viewpoint based on the Liouville geometry. The duality between the topological A-model approach (geometric engineering) and Nekrasov's approach (instanton counting) has been recently studied in [65, 66] microscopically. From our point of view, the essential reason is that the integration over the both moduli spaces is localized when inserted correlators (or the instanton) collide (DKM compactification) and reduces to the multiplication of the lower amplitudes as is typical in the topological gravity.

On the other hand, the duality between the $(4n+2)$- (or n-) punctured spheres and the genus expansion is reminiscent of the relation between the conventional David-Distler-Kawai (DDK) approach [76] and the Liouville F-model approach to the noncritical string theory. In our SYM case, the genus expansion is mapped to the tree-level amplitude (but in the nontrivial background: the insertion of $\frac{1}{a^4}\int d^2zO(z)$ denotes the deformed background), which reminds us of the heterotic-type-II string duality.[24] This must be another example of the reduction mechanism of the higher genus amplitude to the punctured sphere studied in section 3. Furthermore, this analogy enables us to conjecture what corresponds to the graviphoton correction in the noncritical string (genus expansion) formalism. In the punctured sphere approach, we expect that the graviphoton correction comes from the higher genus amplitude, so in the genus expansion approach, we expect that the graviphoton correction comes from the perturbation of the CFT. This should be compared with the 'quantum Riemann surface' approach. It would be very interesting to verify this conjecture, and for this purpose, the graviphoton corrected recursion relation (which should be deeply connected with the higher genus uniformization theory) must be the key step.

At the same time, if we turn the argument around, we have a deep clue to understand the structure of the path integration over the quantum Liouville theory. As we have emphasized, our recursion relation is in the same universality class with the $c=0$ noncritical string theory. Whereas the matrix model solution is known, it is not yet known how to perform the Liouville path integration on the higher genus Riemann surfaces, nor how to integrate over the moduli space in the continuum approach. Since the direct SYM instanton calculation has been performed recently and we have obtained the reduction mechanism in this case, we expect that the similar reduction *should* exist in the case of the $c=0$ noncritical string theory. We believe that a detailed comparison between direct instanton calculation with our integration over the moduli space of punctured spheres will lead us to interesting results.

[24] Another interesting example of this duality is given by the NS-R duality in the $\hat{c}=1$ noncritical string theory [77].

8.1 Connection with Gromov-Witten theory

It is well-known that the genus zero Gromov-Witten theory on the local Hirzebruch surface in the geometric engineering limit determines the prepotential of SU(2) $\mathcal{N} = 2$ SYM theory. Thus it is natural to discuss the connection between our recursion relation and the underlying recursion relation for the Gromov-Witten invariants. Such a recursion relation is derived in [78] by using the underlying WDVV equation. For example, the Gromov-Witten invariants for $\mathbb{P}^2$ satisfy

$$N_d = \sum_{k+l=d} N_k N_l k^2 l \left[l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right], \tag{8.1}$$

for $d \geq 2$, which resembles our bilinear recursion relation in its structure.[25]

To make the physical picture more transparent, take the recursion relation of the Gromov-Witten invariants for $\mathbb{P}^1 \times \mathbb{P}^1$ which is the Hirzebruch surface F_0:[26]

$$\begin{aligned} N_{n,m} = & \sum_{\substack{n_1+n_2=n \\ m_1+m_2=m}} N_{n_1,m_1} N_{n_2,m_2} (n_1 m_2 + n_2 m_1) m_2 \times \\ & \times \left[n_1 \binom{2n+2m-4}{2n_1+2m_1-2} - n_2 \binom{2n+2m-4}{2n_1+2m_1-1} \right]. \end{aligned} \tag{8.2}$$

This recursion relation seems more complicated than ours, which is somewhat related to the fact that we have double expansions — n and m. To relate this recursion relation to ours, we first suppose that a similar recursion relation exists for the local Hirzebruch surface, which has a direct relation to the SW theory. Then in order to derive the SW theory, we need a geometric engineering limit (7.3). In this procedure we take a nontrivial resummation, and the dependence of two parameter reduces to that of one. For the local Hirzebruch surface, in order to reproduce the SW solution, the asymptotic Gromov-Witten invariants $d_{n,m}$ for large m must be given by [33]

$$d_{n,m} \sim \gamma_n m^{4n-3}, \tag{8.3}$$

where γ_n is related to the instanton amplitudes $\mathcal{F}_n$ as

$$\gamma_n = \frac{2^{5n-2}}{(4n-3)!} 2\pi i \mathcal{F}_n. \tag{8.4}$$

From our bilinear recursion relation, we can predict the asymptotic recursion relation of the Gromov-Witten invariants:

$$(4n-4)!\gamma_n = \frac{2^2}{n} \sum_{k=1}^{n-1} g_{k,n} k(n-k)(4k-4)!(4(n-k)-4)! \gamma_k \gamma_{n-k}. \tag{8.5}$$

It is very plausible that our bilinear recursion relation is the remnant of the recursion relation for the Gromov-Witten invariants. It would be an interesting problem to see

[25] In [79], it was noticed that the recursion relation for the N_d might be related to $c = 0$ two dimensional gravity. Actually, the nonperturbative differential equation satisfied by the generating function of N_d is given by the Painlevé VI equation. We also note that this model also can be regarded as a Liouville F-model by using the same technique reviewed in section 3.

[26] The Gromov-Witten invariants for the local $\mathbb{P}^1 \times \mathbb{P}^1$ are *different* from the one for the $\mathbb{P}^1 \times \mathbb{P}^1$. The connection between them is not clear yet, but it should exist as our approach suggests.

whether we can directly obtain our recursion relation in this approach. It is also worth mentioning that our construction of the instanton amplitudes in terms of the rational intersection theory on $\overline{\mathcal{M}}_{0,n}$ yields the asymptotic growth of the Gromov-Witten invariants for the local Hirzebruch surface. This fact indicates that the asymptotic form of the Gromov-Witten invariants is described by the rational intersection theory on $\overline{\mathcal{M}}_{0,n}$. We believe some interesting mathematical structure might be hidden here.

8.2 Graviphoton correction

We have seen that the SW solution in the flat background is deeply connected to the uniformization theory of the punctured sphere. In this subsection, we would like to discuss its possible extension to the graviphoton background. The full prepotential under the graviphoton background has been proposed recently by Nekrasov [10, 11]. We may also calculate the graviphoton background prepotential from the geometric engineering approach as in section 7.1. The direct evaluation of the higher genus Gromov-Witten invariants is hopeless unless we use the mirror symmetry, but fortunately, by using the geometric transition method, the prepotential is calculated as the Chern-Simons theory, which is now refined as the topological vertex method [80].

Another interesting ingredient in the graviphoton background $\mathcal{N}=2$ SYM theory is the recent discovery by [23] that if we properly redefine the modulus $u = \mathrm{Tr}\,\tilde{\phi}^2$, the $\mathcal{F}$-u relation remains the same even in presence of nontrivial graviphoton background. This result would suggested considering the graviphoton corrected PF equations. One may expect bilinear recursion relations that should also shed lights on the underlying algebraic geometrical structure.

Understanding the monodromy properties of the corrected prepotential should lead to a generalization of the uniformization picture observed in SW theory [2, 4]. The relevance of the uniformization theory in the $\mathcal{N}=2$ SYM theory is two-fold. We first observe that any inverse of the uniformization map $\tau(u) = J^{-1}(u)$ defines a physical coupling constant in the moduli space u with a suitable monodromy. This is because τ is a univalent function, and is defined on the upper-half plane.[27] Actually, we can derive the SW solution from the uniformization theory of the thrice punctured sphere. The second one is related to the Liouville F-models which we have heavily used to understand the origin of the genus zero recursion relation.

Let us begin with the geometric engineering setup we have proposed in section 7.1. When we consider the higher genus (hence graviphoton) corrections to the prepotential, we naturally expect from our construction that the free-energy should behave as

$$\mathcal{F}_n^{(g)} \sim \int_{\overline{\mathcal{M}}_{g,n+2}} {\omega_{g,n+2}}^{n-2} \wedge \omega_I \sim \left\langle \int_{\overline{\mathcal{M}}_g} dm \left(\int d^2 z O(z) \right)^n \right\rangle_{S_\infty, g}. \tag{8.6}$$

[27] The fact that positivity of the coupling constant may lead to univalence of coupling constants for effective theories seems to be at the origin of the underlying stringy nature. Actually, the inverse of the uniformizing map has a basic property in the theory of Riemann surfaces. This is strictly related to the universal Teichmüller space $T(1)$ where a nonperturbative formulation of string theory should be formulated.

We have used the assumption that $O(z)$ is almost BRST exact and the contribution of the correlator comes only from the boundary of the moduli spaces. This is guaranteed by the Liouville F-model like construction here and the Wolpert restriction phenomenon. If we evaluate this integral by using the similar technique as we have employed in section 7.2 (note that we need higher genus calculations leading to the genus expansion), we obtain the recursion relation which relates the higher genus amplitude to the lower genus amplitudes. This is expected from the topological string theory: we have a holomorphic anomaly equation [69] which connects higher genus amplitude with the lower genus ones on one hand, and on the other hand we know that the different genus contributions are combined in the Gopakumar-Vafa invariant form [67] before taking the geometric engineering limit.

The other possibility is that we consider the *quantum* uniformization theory on the thrice punctured Riemann surface. This nicely fits the existence of the quantum corrected PF equation. Also, if one considers the geometric engineering and performs a mirror symmetry, the target Kodaira-Spencer theory in the mirror B-model is locally described by that of the Riemann surface. Since quantum Kodaira-Spencer theory provides the genus effect, the concept of the quantum uniformization must occur naturally. Our quantum uniformization theory should involve the 'quantum' Liouville theory in contrast to the classical Liouville theory we have mainly utilized in this paper. In the quantum Liouville theory, the quantum correction parameter b, which is related to the central charge as $c = 1 + 6(b + b^{-1})^2$, is identified with the Planck constant $\hbar$ which is the graviphoton correction. It would be an interesting problem to check whether this conjecture is true or not and what is the actual uniformization geometry.

There exists an interesting possibility that the recent progress in string and SYM theories may lead to a deeper understanding of the uniformization theory, in particular on the theory of modular functions. To see this, let us first consider the genus one correction to the SW prepotential. This is given in terms of the η function [81]

$$\mathcal{F}^{(1)} = -\ln \eta(\tau)\,. \tag{8.7}$$

Now note that the η function has a well-defined $\mathrm{SL}(2,\mathbb{Z})$ monodromy. On the other hand, such a monodromy is the uniformizing group of the sphere with three singularities: one puncture at ∞ and elliptic fixed points of orders 2 and 3. Thus we see that whereas the genus zero prepotential is related to the thrice punctured sphere, in the case of the genus one contribution there appears the uniformizing group for the sphere with elliptic points.[28]

What about the underlying geometry of the higher genus contributions? There are some interesting suggestions. First, we saw that the instanton moduli space is strictly related to the geometry of $\overline{\mathcal{M}}_{0,n}$ and of the DKM stable compactification. On the other hand, we saw that there is a natural mapping between Hurwitz spaces and $\overline{\mathcal{M}}_{0,n}$. It should be also noted that if the higher genus contributions were related to the uniformization

[28] We note that this reflects the monodromy group. In particular, whereas in the case of the thrice punctured sphere the second derivative of the prepotential with respect to a is the inverse of the uniformizing map, in genus one there is still a well defined monodromy, that now is $\mathrm{SL}(2,\mathbb{Z})$, for the prepotential. In fact the relation between the uniformizing map and the prepotential is different with respect to the case of genus zero.

of Riemann surfaces, this would provide a tool to obtain exact results. However, the uniformization theory is a long-standing problem and hense one should expect that new insights should involve particular situations although not yet discovered.

A trivial example in which the uniformization has been solved is when the punctures are at the root of the unity: the constraints on the accessory parameters are sufficient to fix them. In this context, the possible geometry related to the higher genus contributions to the prepotential, may be the one of surfaces which are branched covering. Such surfaces have a high symmetry so it may happen that they are 'exactly solvable'. Furthermore, recently it has been shown that in the case of branched covering of the torus one may obtain explicit solutions for the eigenfunctions of the laplacian [82]. Therefore, even if uniformization and spectra on Riemann surfaces are apparently technically very difficult to solve, there are highly symmetric cases in which these problems have been understood. On the other hand, there are surfaces which are coverings of lower genus Riemann surfaces which arise in matrix model theory, see for example [83]. In the case of the branched covering of the torus, the high symmetry of these particular surfaces reflects in a sort of Dirac constraint of the Riemann period matrix Ω_{jk} [34, 82][29]

$$m'_j - \sum_{k=1}^{g} \Omega_{jk} n'_k = \bar{c} \left(m_j - \sum_{k=1}^{g} \Omega_{jk} n_k \right) , \tag{8.8}$$

where m_j, n_j, m'_j, n'_j are integers. Remarkably, this condition naturally appears in string theory [86], suggesting that more generally there is a selection of the geometry contributing to the genus expansion. It is interesting to observe that the above structures are related to the properties of the abelian differentials and to the theory of quantum billiards [87].

Since the basic underlying group for the prepotential is $\mathrm{SL}(2,\mathbb{Z})$, and considering that in genus zero we have the uniformizing group $\Gamma(2) \subset \mathrm{SL}(2,\mathbb{Z})$, one should investigate whether the higher symmetry we discussed reflects in fixing some specific subgroups of $\mathrm{SL}(2,\mathbb{Z})$ as the monodromy groups of the higher genus contributions to the prepotential. Therefore we should have the sequence of monodromy (uniformizing) groups

$$\Gamma_0 = \Gamma(2) \subset \mathrm{SL}(2,\mathbb{Z}) \qquad \Gamma_1 = \mathrm{SL}(2,\mathbb{Z}) \qquad \Gamma_2 \subset \mathrm{SL}(2,\mathbb{Z}) \qquad \Gamma_3 \subset \mathrm{SL}(2,\mathbb{Z}) \ \ldots , \tag{8.9}$$

with the generic Γ_g subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

There is a related intriguing structure which needs to be mentioned. Namely, the thrice punctured sphere in $\mathcal{N}=2$ instanton theory with gauge group $\mathrm{SU}(2)$ also appears for the *same* group and in the *same* context, but now for the classical theory. More precisely, it turns out [88] that the self-dual Yang-Mills (SDYM) equation

$$F = *F , \tag{8.10}$$

[29] It would be very interesting to understand the Schottky problem for such period matrices. Presumably these number theoretical conditions, which leads to complex multiplication (CM) for the jacobian [82], should imply some identity related to the Fay trisecant identity (see for example [84]). This would be of interest in considering the moduli integration in string theory, where the Fay trisecant identity, as first observed by Eguchi and Ooguri [85], corresponds to the bosonization formula.

for the gauge group given by the volume preserving diffeomorphisms of SU(2), has solutions parametrized just by the uniformizing equation for the thrice punctured sphere! Remarkably, there are two interesting solutions, one is described by the Hauptmodule, i.e. inverse of the uniformizing map, for $H/\mathrm{SL}(2,\mathbb{Z})$, while the other concerns $H/\Gamma(2)$, exactly the first two groups arising in the genus zero and one contribution to the prepotential.[30] Interestingly, while the genus one prepotential in $\mathcal{N}=2$ SYM concerns the graviphoton corrections, the self-dual reduction considered in [88] concerns the Bianchi IX cosmological model (see also [89] for the appearence of such uniformizing equations in other related contexts). On the other hand the appearance of volume preserving diffeomorphisms of SU(2) just indicates the emergence of gravity. We believe that this promising analogy deserves to be further understood. A first interesting question is to understand whether the generalization to higher rank groups of such 'gravitational SDYM equations' are in turn similarly related to the higher rank PF equations of SW theory.

8.3 Higher rank gauge groups

Here we would like to discuss another possible extension of our results, namely to the higher rank gauge groups [90]. First of all, we note that the general expression corresponding to the $\mathcal{F}$-u relation (6.25) for the higher rank $\mathcal{N}=2$ SYM theory was derived in [91]. This relation should be the basis of our construction. We also note that similar relations exist between the higher rank group invariants [12, 63].

Our construction of the SU(2) gauge group is based on the Liouville F-models, which provides a certain universality class of the string theory including the $c=0$ quantum gravity. Actually, we have pointed out many similarities between the SU(2) SW theory and the $c=0$ noncritical string theory in this paper. Therefore it is natural to relate the higher rank gauge groups SW theory to the ADE minimal models coupled to the two-dimensional quantum gravity (Liouville theory). In particular, SU(N) gauge theory is related to the A_{N-1} minimal models. Of course, SU(2) is given by the A_1 minimal model coupled to the gravity, or $c=0$ Liouville theory itself.

The emergence of the ADE minimal models is also expected from the geometric engineering approach. In the case of the ADE gauge group, the fiber of the base $\mathbb{P}^1$ is given by the surface which possesses the ADE singularity in the geometric engineering limit. At the same time, it is well-known that the ADE singularity in its local form is described by the ADE $\mathcal{N}=2$ minimal models. If we twist the theory, $\mathcal{N}=2$ minimal model is supposed to become the *bosonic* (or topological) minimal model, which we couple to the two-dimensional (topological) gravity.

From the duality to the matrix model, we can derive the nonlinear recursion relation for the free-energy of the unitary minimal models coupled to the two-dimensional gravity. The deformation parameter there should be related to the moduli parameter of the $\mathcal{N}=2$ SYM theory. As in the SU(2) case which we have thoroughly studied in this paper, we will

[30] In [88] it has been obtained a differential equation relating the two Hauptmodules based on the well known relation. These relations imply a connection between the genus zero and genus one contributions to the prepotential. It would be interesting to study such equations to gain insights on the existing relations between different genus contributions to the prepotential. This would relate the monodromies.

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be able to obtain the recursion relation for the SW theory by deforming the underlying recursion relation for the ADE minimal gravity.

Finally we should add that in [55], it was realized that the unitary minimal models coupled to the two-dimensional gravity can be formulated in a purely geometrical manner. Therefore, there is a possibility that the $\mathcal{N}=2$ SYM theory with an arbitrary gauge group can be described solely in terms of the quantum geometry of the Riemann surface. This would be an interesting problem worth studying further.

9. Conclusion

In this paper, we have studied the Liouville geometry of the $\mathcal{N}=2$, SU(2) SYM theory and proposed a bilinear recursion relation based on the observation that the theory has a similar structure with the Liouville geometry of the punctured sphere. By utilizing the underlying Liouville theory, we succeeded in presenting the physical origin of the bilinear recursion relation in three different ways.

While these expressions have a firm ground from the macroscopic point of view, it remains a very interesting problem to derive them from the more microscopic point of view. Let us state three major microscopic problems here to conclude the whole paper and indicate the future directions.

Firstly, from our study we believe in the existence of the direct map between the ADHM moduli space and the moduli space of the punctured spheres, and the connection between the various localization formula and our boundary of the moduli space. Thus the rederivation of our results from the direct instanton calculation along the recent developments around Nekrasov's complete solution should cast a new insight into the structure of the instanton moduli space. Secondly, in the geometric engineering approach, the derivation of the bilinear recursion relation as a consequence of the underlying relation for the Gromov-Witten invariants, whose possibility we have sketched in section 8.2 is another intriguing challenge from the microscopic perspective. Finally, in the noncritical string approach, the central problem is to find a microscopic action for the theory, whose discovery and understanding of its reduction mechanism should also bring a new impact on the evaluation of the Liouville path integral from the first principle. We would like to report the progress on these fascinating subjects in the near future.

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A. The restriction phenomenon and the Weil-Petersson divisor

A.1 Wolpert's theorem

Let us start by proving the restriction phenomenon. Namely we show that from the natural embedding

$$i:\ \overline{V}^{(m)} \to \overline{V}^{(m)} \times * \to \overline{V}^{(m)} \times \overline{V}^{(n-m+2)} \to \partial\overline{V}^{(n)} \to \overline{V}^{(n)}\,, \qquad n>m\,,$$

where $*$ is an arbitrary point in $\overline{V}^{(n-m+2)}$, it follows that [39]

$$[\omega_m] = i^*\,[\omega_n]\,, \qquad n>m\,. \tag{A.1}$$

In order to prove (A.1) we need to consider the Fenchel-Nielsen parametrization of the Teichmüller space (see for example [92]). Let $\{P_i\}$ be a set of surfaces homeomorphic to $\widehat{\mathbb{C}}$ minus three open discs. Each P_i has a hyperbolic structure with geodesic boundary whose length may be arbitrarily prescribed in the interval $[0,\infty)$ (a length 0 corresponds to a puncture). Let

$$\Sigma_{0,n} = \mathbb{C}\backslash\{z_1,\ldots,z_{n-3},0,1\}\,, \tag{A.2}$$

be a genus 0 surface with n punctures. It can be obtained by glueing $\{P_i\}_{i=1,\ldots,n-2}$ identifying the different boundary components in $n-3$ geodesics on $\Sigma_{0,n}$. Clearly, to completely characterize the glueing procedure, we need also to distinguish twisted boundary components. To this end, for each geodesic α_i we denote by τ_i the coordinate describing twists from an arbitrary reference position. Denoting by $l_i = l_{\alpha_i}$ the length of each geodesic, we define the Fenchel-Nielsen form

$$\omega_{FN}^{(n)} = \sum_{j=1}^{n-3} dl_j \wedge d\tau_j = \sum_{j=1}^{n-3} l_j dl_j \wedge d\theta_j\,, \tag{A.3}$$

where θ_j is the twisting angle (it has been proved that $\omega_{FN}^{(n)}$ does not depend on the particular geodesic dissection of $\Sigma_{0,n}$).

The observation that the smooth reduction of $\omega_{FN}^{(n)}$ to $\partial\overline{V}^{(n)}$ is performed letting one or more geodesical lengths go to 0 giving a well-defined geodesical dissection, implies $[\omega_{FN}^{(m)}] = i^*[\omega_{FN}^{(n)}]$, $n>m$. Eq. (A.1) follows by noticing that [39]

$$[\omega_{FN}^{(n)}] = [\omega_n]\,, \tag{A.4}$$

in $H^2\left(\overline{V}^{(n)},\mathbb{R}\right)$.

A.2 Moving puncture and $D_{WP}^{(n)}$

Now, following [38], we show that

$$D_{WP}^{(n)} = \frac{\pi^2}{n-1}\sum_{k=1}^{[n/2]-1} k(n-k-2)D_k\,. \tag{A.5}$$

Observe that the coordinate of a Riemann surface can be seen as a moving puncture. Therefore, we can consider the embedding of $\Sigma_{0,n-k}$ in $V^{(n-k+1)}$

$$\Sigma_{0,n-k} \longrightarrow V^{(n-k+1)}\,, \qquad z \mapsto (z_1, \ldots, z_{n-k-3}, z) \in V^{(n-k+1)}\,, \quad z \in \Sigma_{0,n-k}\,. \tag{A.6}$$

Observe that $\Sigma_{0,n-1}$ embeds into $V^{(n)}$ and therefore also into $\overline{V}^{(n)}$. A natural embedding into $\overline{V}^{(n)}$ can be defined also for the surfaces $\Sigma_{0,n-k}$, $k = 2, \ldots, [n/2] - 1$, namely

$$\Sigma_{0,n-k} \to V^{(n-k+1)} \to \overline{V}^{(k+1)} \times \overline{V}^{(n-k+1)} \to D_{k-1} \to \overline{V}^{(n)}\,, \qquad k = 2, \ldots, \left[\frac{n}{2}\right] - 1\,. \tag{A.7}$$

The closure of the image of $\Sigma_{0,n-k}$ in $\overline{V}^{(n)}$ defines a 2-cycle C_k isomorphic to $\widehat{\mathbb{C}}$. By (A.1) and (A.7) it follows that

$$[\omega_n] \cap [C_k] = \int_{i\Sigma_{0,n-k}} \omega_n = \int_{\Sigma_{0,n-k}} i^*\omega_n = \int_{\Sigma_{0,n-k}} \omega_{n-k+1}\,, \tag{A.8}$$

where $\cap$ denotes the topological cap product. Note that $[\omega_n] \cap [C_k] = D_{WP}^{(n)} \cdot C_k$ where $\cdot$ denotes the topological intersection (see for example [93]). In order to perform the last integral we use (2.14) and the asymptotic behavior of the classical Liouville action when the punctures coalesce [94]

$$\partial_{z_i} S_{cl}^{(n)}(z_1, \ldots, z_{n-3}) = \begin{cases} \dfrac{\pi}{z_i - z_k} + o\left(\dfrac{1}{|z_i - z_k|}\right); & z_i \to z_k\,, \qquad k \neq n\,; \\ \dfrac{\pi}{z_i} + o\left(\dfrac{1}{|z_i|}\right); & z_i \to \infty\,. \end{cases} \tag{A.9}$$

Now observe that[31]

$$D_{WP}^{(n)} \cdot C_k = \int_{\mathbb{C}} \omega_{n-k+1} = -\frac{1}{2i} \lim_{r \to 0} \int_{\mathbb{C}\backslash\Delta_r} d\partial_{z_{n-k-2}} S_{cl}^{(n-k+1)} dz_{n-k-2}\,, \tag{A.10}$$

where Δ_r is the union of $n - k - 1$ disks of radius r centered at $z_1, \ldots, z_{n-k-3}, 0, 1$. Let us now set $z \equiv z_{n-k-2}$. Since $\partial_z S_{cl}^{(n-k+1)} \in \mathcal{C}^\infty(\mathbb{C}\backslash\Delta_r)$, we can apply Stokes theorem

$$\begin{aligned} \int_{\mathbb{C}\backslash\Delta_r} d\partial_z S_{cl}^{(n-k+1)} dz &= \int_{\partial\mathbb{C}} \partial_z S_{cl}^{(n-k+1)} dz - \int_{\partial\Delta_r} \partial_z S_{cl}^{(n-k+1)} dz \\ &= 2i\pi^2 - 2i\pi^2(n - k - 1)\,. \end{aligned} \tag{A.11}$$

On the other hand

$$\lim_{r \to 0} \int_{\Delta_r} d\partial_z S_{cl}^{(n-k+1)} dz = 0\,, \tag{A.12}$$

so that

$$D_{WP}^{(n)} \cdot C_k = \pi^2(n - k - 2)\,. \tag{A.13}$$

[31]The integrals are understood in the sense of Lebesgue measure.

Eq. (A.5) follows immediately from (A.13) and from the non singular matrix $A_{jk} = C_j \cdot D_k$ of intersection numbers between the 2-cycles C_j and the $(2n-8)$-cycles D_k [38]

$$A = \begin{pmatrix} n-1 & 0 & 0 & 0 & \dots & 0 & 0 \\ n-4 & 1 & 0 & 0 & \dots & 0 & 0 \\ n-4 & -1 & 1 & 0 & \dots & 0 & 0 \\ n-5 & 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-[n/2] & 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}, \qquad \text{(A.14)}$$

where $C_j \cdot D_1 = n - j - 1$ for $j \geq 4$.

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